# Phys 251A Problem Set 9
## Date posted: November 15, 2024
## Due date: November 22, 2024

1. Consider a perturbed Harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{g}{l^3}x^3$$

where $g \ll \omega$ has the dimensions of energy and $l = \sqrt{\hbar/m\omega}$ is the oscillator length. You should treat the term proportional to $g$ perturbatively.

   (a) Express the above problem in terms of creation and annihilation operators (this will make subsequent calculations easier). Show that to first order in $g$ the ground state energy shift is zero. Calculate the shift to second order in $g$.

   (b) Calculate the perturbed wavefunction to leading order in $g$.

   (c) Sketch the potential $V(x)$ in the above Hamiltonian for some small, nonzero $g$, and, on top of it, sketch the perturbed wavefunction to leading order in $g$. Does perturbation theory capture the true ground state? What physical process does perturbation theory not take into account? [If you prefer to type your problem set and don't want to make a figure, a sufficiently vivid description will suffice here]

   (d) Let us consider the perturbation $\frac{g}{l^4}x^4$ instead. This term is typically present in symmetric potential wells, and leads to unequal spacing between energy levels ("anharmonic oscillator"). By sketching the total potential for small but nonzero $g > 0$ and $g < 0$, argue that a power series expansion of the ground state energy around $g = 0$ has zero radius of convergence.

   *Here we are using the fact that any power series $\sum_n c_n x^n$ has a radius of convergence $R \geq 0$ such that the series converges for $|x| < R$ but diverges for $|x| > R$. Note that series with $R = 0$ are not useless, they can be regarded as asymptotic series. A good approximation can be obtained by keeping terms until they start to increase in magnitude. Sometimes advanced resummation techniques can be used to extract a convergent result (Borel summation)*

2. A perturbed three dimensional harmonic oscilator: consider the Hamiltonian

$$H = \hbar\omega(N_1 + N_2 + N_3 + \frac{3}{2}) + \kappa L_3$$

where $N_i = a_i^\dagger a_i$ are the number operatorss of the $i$'th oscillator and $L_3 = x_1 p_2 - x_2 p_1$ is the $z$ component of angular momentum. The unperturbed Hamiltonian, $H_0$, is $H$ evaluated at $\kappa = 0$.

   (a) Let us label the unperturbed states as $|n_1, n_2, n_3\rangle$ where $n_i$ is the eigenvalue of $N_i$. How many linearly independent states are there with energy $\frac{5}{2}\hbar\omega$ for $\kappa = 0$?

   (b) Derive an effective Hamiltonian, within the degenerate subspace you described in the previous part, up to first order in $\kappa$, and find its energy eigenvalues and eigenvectors.

   (c) Write the eigenstates of $H$, with energies near $\frac{5}{2}\hbar\omega$, in terms of linear combinations of $|n_1 n_2 n_3\rangle$ to first order in $\kappa$.

   (d) Are there corrections beyond first order in $\kappa$ in this case? Why or why not? Hint: what is $[H_0, L_3]$?