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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 14: First Hourly (Practice A)

Problems

Problem 14A.1 (10 points):
The **Fibonacci numbers** are defined recursively as follows: start with $F_0 = 0, F_1 = 1$ then define $F_{n+1} = F_n + F_{n-1}$, so that $F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ etc. Prove that

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$$

for every positive integer n.

Solution:
This is definitely a statement we can prove **by induction**.
(i) We first check the **induction foundation** which is the case $n = 1$.
Indeed, we have $1 = F_3 - 1 = 1$.
(ii) Now we check the **induction step**: assume the statement is true for n. We have to check that the statement holds for $n + 1$. So, lets prove

$$F_0 + \cdots + F_{n+1} = F_{n+3} - 1 \ .$$

The left hand side of this equation can be written, using the induction assumption, as $(F_{n+2} - 1) + F_{n+1}$. This is also $F_{n+1} + F_{n+2} - 1$. By using the definition of the Fibonacci sequence, this is $F_{n+3} - 1$ which agrees with the right hand side and so finishes the proof.
Q uite E asily D one.

Problem 14A.2 (10 points):
Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

a) (4 points) Compute AB and $\text{rref}(AB)$.
b) (4 points) Now row reduce both A and B and form $\text{rref}(A)\text{rref}(B)$.
c) (2 points) Is the statement $\text{rref}(AB) = \text{rref}(A)\text{rref}(B)$ true for all A, B?

Solution:
a) The product is $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. After row reduction we have $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.
b) We have $\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathrm{rref}(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The product is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
c) No, as a) and b) show, they are not equal in general. Row reduction is not compatible with matrix multiplication. By the way, for square matrices, the equation is almost always true as if we chose a matrix randomly, then almost certainly, it is invertible so that the row reduced echelon form is the identity matrix.

Problem 14A.3 (10 points):
a) (2 points) Parametrize the line through $(1, 1, 1)$ and $(4, 3, 1)$ in $\mathbb{R}^3$.
b) (2 points) Parametrize the ellipse $x^2/16 + y^2/25 = 1$ in $\mathbb{R}^2$.
c) (2 points) Parametrize the graph $y = x^5 + x$ in $\mathbb{R}^2$.
d) (2 points) Parametrize the circle $x^2 + (y - 2)^2 = 1, z = 4$ in $\mathbb{R}^3$.
e) (2 points) Parametrize the line $x = y = z$ in $\mathbb{R}^3$.

Solution:
a) $r(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$.
b) $r(t) = \begin{bmatrix} 4\cos(t) \\ 5\sin(t) \end{bmatrix}$.
c) $r(t) = \begin{bmatrix} t \\ t^5 + t \end{bmatrix}$.
d) $r(t) = \begin{bmatrix} \cos(t) \\ 2 + \sin(t) \\ 4 \end{bmatrix}$.
e) $r(t) = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$.

Problem 14A.4 (10 points):
Find the arc length of the curve

$$r(t) = [t\cos(t^2), t\sin(t^2), t^2]$$

for $0 \le t \le 2$.

Solution:
We compute first $r'(t) = [\cos(t^2) - 2t^2\sin(t^2), \sin(t^2) + 2t\cos(t^2), 2t]$. Its length is after some simplification (you have to include details of your computations!) $|r'(t)| = 1 + 2t^2$. Integrate this from 0 to 2 is

$$\int_0^2 1 + 2t^2 \; dt = t + 3t^3/3|_0^2 = 22/3 \; .$$

Problem 14A.5 (10 points):
a) (2 points) What is the Heine-Cantor theorem?
b) (2 points) Formulate the triangle inequality.
c) (2 points) What is the Al Kashi identity?
d) (2 points) Give the name of a nowhere differentiable function.
e) (2 points) Is it true that a continuous curve $r(t)$ has a finite arc length?

Solution:
a) On a closed interval, a function is uniformly continuous if and only if it is continuous.
b) If a, b, c are the length of a triangle, then $c \leq a + b$.
c) $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.
d) The Weierstrass function.
e) The Koch curve was a counter example.

Problem 14A.6 (10 points):
a) (2 points) Find $(3 + i)(4 + 2i)$
b) (2 points) What is $e^{i3\pi/4}$?
c) (2 points) Convert from cylindrical $(r, \theta, z) = (2, \pi/2, 1)$ to Cartesian.
d) (2 points) What are the spherical coordinates of $(1, \sqrt{3}, 2)$?
e) (2 points) What surface is in spherical coordinates given as $\rho\sin(\phi) = 1$?

Solution:
a) $10 + 10i$.
b) $-1/\sqrt{2} + i/\sqrt{2}$.
c) $(0, 2, 1)$
d) $(\rho, \theta, \phi) = (\sqrt{8}, \pi/3, \pi/4)$.
e) $r = 1$. A cylinder.

Problem 14A.7 (10 points):

a) (5 points) You are given $r'''(t) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $r(0) = (7, 8, 9)$ and $r'(0) = (1, 0, 0)$ and $r''(0) = (0, 1, 0)$. Find $r(1)$.

b) (5 points) What is the curvature of $r(t) = [t, t+t^2, t+t^2+t^3]$ at $t = 0$?

Solution:

We write here $[a, b, c]$ for a vector even so we actually deal with a column vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

On paper, it is better to use column vectors. Here we want to save some space and because writing column vectors is a bit more work in typed text. The advantage to stick to the columns is that the Jacobian matrix makes more sense then.

a) $r(t) = [7, 8, 9] + [t, 0, 0] + [0, t^2/2, 0] + [3t^3/3, 4t^3/3, 5t^3/3]$. $r(1) = [17/2, \frac{55}{6}, \frac{59}{6}]$.

b) $r' = [1, 1, 1], r''(0) = [0, 2, 2]$.

The cross product $r' \times r'' = [0, -2, 2]$ has length $\sqrt{8}$. The curvature formula gives $\sqrt{8}/(3\sqrt{3})$.

Problem 14A.8 (10 points):

a) (5 points) Find a parametrization $r(u, v)$ of the cylinder $x^2 + z^2 = 9$.

b) (5 points) Find $r(u, v)$ for the paraboloid $y^2 + 3z^2 = x$.

Solution:

Surface a) is a surface of revolution. Surface b) is a graph of a function, but we express x as a function of y and z and not as usual z as a function of x and y.

a) $r(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 3\cos(u) \\ v \\ 3\sin(u) \end{bmatrix}$.

b) $r(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} u^2 + 3v^2 \\ u \\ v \end{bmatrix}$.

Problem 14A.9 (10 points):

Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$. a) (2 points) The image of A is a plane. By using the cross product, write it as $ax + by + cz = d$.
b) (2 points) What is the first fundamental form $g = A^T A$?
c) (2 points) From a) you have $[a, b, c]^T = v \times w$. Find $\sqrt{a^2 + b^2 + c^2}$.
d) (2 points) Find the distortion factor $||A|| = \sqrt{\det(A^T A)}$ of A.
e) (2 points) What theorem was involved to see $||A|| = |v \times w|$?

Solution:
a) The normal vector n is the kernel of A^T. We could row reduce to find this vector but it is easier to use the cross product in three dimensions as the cross product is perpendicular to the two column vectors of A. The cross product is which is $n = [a, b, c] = [1, 0, -1]^T$. The equation is $ax + by + cz = d$ which is $x - z = 0$. Note that d is zero because the plane contains the point $(0, 0, 0)$.
b) This is just matrix multiplication $A^T A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 3 \end{bmatrix}$. (note that we want to see such things written out in your solution.)
c) The length of n is $\sqrt{1 + 0 + 1} = \sqrt{2}$.
d) The determinant of g is $6 * 3 - 4 * 4 = 2$. The distortion factor is $\sqrt{\det(g)} = \sqrt{2}$.
d) This was a consequence of the Cauchy-Binet formula. It was one of the theorems we have proven.

Problem 14A.10 (10 points):
a) (5 points) What is the Jacobian matrix df of the map

$$f(x, y, z) = [x^2 + y^2 + z^2, x + y, -x^2]^T \ ?$$

b) (5 points) Find the distortion factor $\det(df)$.

Solution:
a) To compute the Jacobian, take in the first column the derivative with respect to x, then in the second column, the derivative with respect to y and in the third column the derivative with respect to z.

$$df = \begin{bmatrix} 2x & 2y & 2z \\ 1 & 1 & 0 \\ -2x & 0 & 0 \end{bmatrix} .$$

b) The determinant of df is $4xz$. So far, we have only looked at determinants of 2×2 and 3×3 matrices. In the 3×3 case, you can see it as the triple scalar product of the three column vectors. Note that each of the column vectors of df is a velocity vector of the curve, if you just change one of the variables and keep the others constant.