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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 14: Hourly 1 (Practice B)

Problems

Problem 14B.1 (10 points):
Prove that $1+2+4+8+\cdots+2^n = 2^{n+1}-1$ for every positive integer n.

Solution:
(i) the induction assumption for $n=1$ gives $3=3$ on both sides.
(ii) the induction step: assume $1+2+4+8+\cdots+2^n = 2^{n+1}-1$. using this assumption we have $1+2+4+8+\cdots+2^{n+1} = (1+2+4+8+\cdots+2^n)+2^{n+1} = 2^{n+1}-1+2^{n+1}$. The right hand side simplifies to $2^{n+2}-1$ which is the right hand side of the statement in the case $n+1$.

Problem 14B.2 (10 points):

a) (5 points) Row reduce the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$.

b) (5 points) Compute the matrix product $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution:

$\text{rref}(A) == \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) The matrix product gives 122.

Problem 14B.3 (10 points):
a) (2 points) Parametrize the curve $x = \sin(y)$ in $\mathbb{R}^2$.
b) (2 points) Parametrize the curve $r = \sin^2(5\theta)$ in $\mathbb{R}^2$.
c) (2 points) Parametrize the curve $y = x^5 + x, z = 4$ in $\mathbb{R}^3$.
d) (2 points) Parametrize the line $2x + y = 4$ in $\mathbb{R}^2$.
e) (2 points) Parametrize the ellipse $(x-1)^2 + \frac{y^2}{4} = 1$ in $\mathbb{R}^2$.

Solution:
a) $r(t) = \begin{bmatrix} \sin(t) \\ t \end{bmatrix}$.
b) $r(t) = \begin{bmatrix} \sin^2(5t)\cos(t) \\ \sin^2(5t)\sin(t) \end{bmatrix}$.
c) $r(t) = \begin{bmatrix} t \\ t^5 + t \\ 4 \end{bmatrix}$.
d) $r(t) = \begin{bmatrix} t \\ 4 - 2t \end{bmatrix}$.
e) $r(t) = \begin{bmatrix} \cos(t) + 1 \\ 2\sin(t) \end{bmatrix}$.

Problem 14B.4 (10 points):
Find the arc length of the curve

$$r(t) = \begin{bmatrix} e^t \\ e^{-t} \\ \sqrt{2}t \end{bmatrix}$$

for $0 \leq t \leq 1$.

Solution:
We are led to the integral $\int_0^1 \sqrt{e^{2t} + e^{-2t} + 2}\, dt$. The part in the square root is a square $(e^t + e^{-t})^2$. Now the integral is no problem: $e - 1/e$.

Problem 14B.5 (10 points):
a) (2 points) Formulate the Cauchy-Schwarz inequality.
b) (2 points) What formula gives the area of the parallelogram spanned by two vectors v and w?
c) (2 points) What formula gives the volume of a parallelepiped spanned by three vectors u, v, w?
d) (2 points) Who invented the quaternions?
e) (2 points) Assume $\text{rref}(A) = \text{rref}(B)$. Does this mean $A = B$?

Solution:
a) $|v \cdot w| \leq |v||w|$
b) $|v \times w$
c) $|u \cdot (v \times w)|$
d) Hamilton. Here is a list of answers which the creative class made up: none of the following are typos. This is how the appeared: Liepzig, Hidleberg, Descartes, Descaste, Newton, Heine, Bineet, Some man, Cantor, Cauchy, Wierness, Gauss, Euler and Boltzmann. Newton got even three votes.
e) No, for example, all diagonal matrices with nonzero diagonal entries row reduce to the identity matrix.

Problem 14B.6 (10 points):
a) (2 points) Write the complex number $z = e^{-i\pi/2}$ in the form $z = a + ib$.
b) (2 points) Which point (x, y, z) has the cylindrical coordinates $(r, \theta, z) = (1, \pi/2, 0)$?
c) (2 points) What are the spherical coordinates (ρ, ϕ, θ) of the point $(x, y, z) = (\sqrt{2}, \sqrt{2}, -2)$?
d) (2 points) What surface is $\rho \sin^2(\phi) = \cos(\phi)$? Give the name and write it in Cartesian coordinates
e) (2 points) What surface is given in cylindrical coordinates by the equation $r \sin(\theta) = 2$?

Solution:
a) $-i$
b) $(0, 1, 0)$
c) $(\sqrt{8}, 3\pi/4, \pi/4)$
d) Multiply with ρ to see $r^2 = z$, a paraboloid.
e) $y = 2$ is a plane.

Problem 14B.7 (10 points):

a) (5 points) You are given $r''(t) = \begin{bmatrix} 0 \\ 3 \\ t \end{bmatrix}$ and $r(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $r'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Find $r(1)$.

b) (5 points) What is the curvature of $r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$ at $t = 0$?

Solution:
a) $r(t) = [t, 3t^2/2, t^3/6]^T$ which is $(1, 3/2, 1/6)$ at $t = 1$.
b) The curvature is $\sqrt{2}/\sqrt{2}^3 = 1/2$.

Problem 14B.8 (10 points):
a) (2 points) Find a parametrization of the cone $x^2 + y^2 = z^2$.
b) (2 points) Find a parametrization of $x^2/4 + y^2/9 + z^2/16 = 1$.
c) (2 points) Find a parametrization of the surface $x^2 - y^2 = z$.
d) (2 points) Find a parametrization of the plane $z = 2$.
e) (2 points) Find a parametrization of the cylinder $x^2 + z^2 = 1$.

Solution:

a) $\begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} z\cos(\theta) \\ z\sin(\theta) \\ z \end{bmatrix}$.

b) $\begin{bmatrix} \phi \\ \theta \end{bmatrix} = \begin{bmatrix} 2\sin(\phi)\cos(\theta) \\ 3\sin(\phi)\sin(\theta) \\ 4\cos(\phi) \end{bmatrix}$.

c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ x^2 - y^2 \end{bmatrix}$.

d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 2 \end{bmatrix}$.

e) $\begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ y \\ \sin(\theta) \end{bmatrix}$.

Problem 14B.9 (10 points):
a) (5 points) Find the dot product $A \cdot B = \text{tr}(A^T B)$ between the two matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

b) (5 points) Find the cosine of the angle between these two matrices.

Solution:
a) 3.
b) $\cos(\alpha) = 3/(\sqrt{6}\sqrt{3}) = 1/\sqrt{2}$.

Problem 14B.10 (10 points):
a) (5 points) What is the Jacobian matrix df of the coordinate change

$$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - y + \sin(x) \\ x \end{bmatrix}.$$

b) (5 points) What is the distortion factor $\det(df)$ of the map f which by the way is called the **Chirikov map**.

Solution:
a) $\begin{bmatrix} 2 + \cos(x) & -1 \\ 1 & 0 \end{bmatrix}$.
b) The determinant is 1.