LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 14: Keywords for First Hourly

This is a bit of a checklist. Make your own list. But here is a checklist which tries to be comprehensive. Check off the topics you know and check back with things you do not recall. You will need to have the following on your finger tips.

Theorems

- [ ] Cauchy-Schwarz $|v \cdot w| \leq |v||w|$ in general for $M(n,m)$
- [ ] Pythagoras $c^2 = a^2 + b^2$ for any inner product space
- [ ] Al Khashi $c^2 = a^2 + b^2 - 2ab\cos(\alpha)$ for any triangle
- [ ] Uniqueness of row reduction: $rref(A)$ is unique in $M(n,m)$
- [ ] The dot product formula $v \cdot w = |v||w|\cos(\alpha)$
- [ ] The cross product formula $|v \times w| = |v||w|\sin(\alpha)$
- [ ] Image of transpose $\mathrm{im}(A^T)$ is kernel $\ker(A)$
- [ ] Cauchy-Binet formula $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$
- [ ] Arc length $\int_a^b |r'(t)|\,dt$ for differentiable r
- [ ] Curvature formulas $|T'|/|r'| = |r' \times r''|/|r'|^3$
- [ ] Euler formula $e^{it} = \cos(t) + i\sin(t)$ and special case
- [ ] Distortion formula $\sqrt{\det(dr^T dr)} = |r_u \times r_v|$ for $r : \mathbb{R}^2 \to \mathbb{R}^3$

Proofs

- [ ] The use of precise definitions and notation
- [ ] Be able to argue by contradiction
- [ ] Think visually, make good pictures
- [ ] Use algebra to tackle geometric problems
- [ ] Master the method of induction
- [ ] Know the benefits and risks of intuition
- [ ] Be aware of computer assisted verification
- [ ] Believe in your creativity

Algorithms

- [ ] Find the angle between vectors or matrices
- [ ] Find the area of parallelogram
- [ ] Find the volume of parallelepiped
- [ ] Row reduce a matrix in $M(n,m)$
- [ ] Get position from velocity or acceleration
- [ ] Find the vector perpendicular to a plane

- [ ] Find the length of a curve or matrix
- [ ] Find the curvature at some point
- [ ] Compute with complex numbers
- [ ] Switch between coordinate systems
- [ ] Compute the distortion factor
- [ ] Get distances between objects

Objects

- [ ] Matrices A
- [ ] Column- and row vectors
- [ ] Parametrized curves $r(t) = [x(t), y(t), z(t)]^T$
- [ ] Parametrized surfaces $r(u, v) = [x(u, v), y(u, v), z(u, v)]^T$
- [ ] Functions $f(x, y, z)$.
- [ ] Level surfaces $f(x, y, z) = d$
- [ ] Linear manifolds $\{x|Ax = d\}$
- [ ] Quadratic manifolds $\{x|x^T Bx + Ax = d\}$
- [ ] Kernel of a linear map $\{x|Ax = 0\}$
- [ ] Image of a linear map $\{Ax|x \in \mathbb{R}^n\}$

Differentiation

- [ ] Velocity r'
- [ ] Acceleration r''
- [ ] Jerk r'''
- [ ] Free fall: $r'' = v$ given
- [ ] TNB frame, $T = r'/|r'|, N = T'/|T'|, B = T \times N$
- [ ] derivative $dr \in M(n, m)$ of a map $\mathbb{R}^m \to \mathbb{R}^n$
- [ ] Jacobian matrix dr of a map $\mathbb{R}^n \to \mathbb{R}^n$
- [ ] Distortion factor $\sqrt{\det(dr^T dr)}$
- [ ] Distortion factor for $n = m$ simplifies to $|\det(dr)|$
- [ ] Example: $r'(t) = dr$, $\sqrt{\det(dr^T dr)} = |r'|$ is speed
- [ ] Curvature $|T'|/|r'|$. In $\mathbb{R}^3$ also $|r' \times r''|/|r'|^3$

Integration

- [ ] Integrate to get arc length.
- [ ] Integrate to get position from velocity etc.
- [ ] Integration technique: substitution
- [ ] Integration technique: integration by parts
- [ ] Integration technique: partial fractions
- [ ] Integration technique: simplification

Coordinate systems

- [ ] Cartesian coordinates
- [ ] Polar coordinates
- [ ] Cylindrical coordinates
- [ ] Spherical coordinates

- General coordinate change
- Distortion factor $|\det(dr)| = \sqrt{\det(dr^T dr)}$

Parametrized Surfaces

- Spheres
- Surfaces of revolution
- Graphs
- Planes
- Torus
- Helicoid

People

- Mandelbrot
- Hamilton
- Descartes
- Cauchy
- Binet
- Schwarz
- Euler
- Heine
- Cantor
- Bolzano
- Archimedes
- Newton
- Einstein
- Napoleon

Geometry of Space

- $v = [v_1, v_2, v_3]^T, w = [w_1, w_2, w_3]^T$, $v + w = [v_1 + w_1, v_2 + w_2, v_3 + w_3]^T$
- dot product $v.w = v_1w_1 + v_2w_2 + v_3w_3 = |v||w|\cos(\alpha)$
- angle $\cos(\alpha) = (v \cdot w)/|v||w|$.
- cross product $v \cdot (v \times w) = 0$, $w \cdot (v \times w) = 0$
- area parallelogram $|v \times w| = |v||w|\sin(\alpha)$
- triple scalar product $u \cdot (v \times w)$
- volume of parallelepiped: $|u \cdot (v \times w)|$
- parallel vectors $v \times w = 0$, orthogonal vectors: $v \cdot w = 0$
- scalar projection $\text{comp}_w(v) = v \cdot w/|w|$
- vector projection $\text{proj}_w(v) = (v \cdot w)w/|w|^2$
- completion of square: $x^2 - 4x + y^2 = 1$ gives $(x-2)^2 + y^2 = 5$
- unit vector = direction: vector of length 1.

Lines, Planes, Functions

- parametric equation for plane $r(t, s) = p + tv + sw$ containing p
- plane $A^T[x, y, z] = ax + by + cz = d$
- parametric equation for line $r(t) = p + tv$ containing p

graph $G = \{(x, y, f(x, y)) \mid (x, y)$ in the domain of $f\ \}$
plane $ax + by + cz = d$ has normal $n = [a, b, c]^T$
line $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ contains $v = [a, b, c]^T$
plane through A, B, C: find normal vector $(a, b, c) = AB \times CB$

Level surfaces

intercepts: intersections of a surface with coordinate axis
traces: intersections of a surface with coordinate planes
generalized traces: intersections with $\{x = c\ \}, \{y = c\ \}$ or $\{z = c\ \}$
level surface $g(x, y, z) = c$: Example: graph $g(x, y, z) = z - f(x, y)$
linear equation like $2x + 3y + 5z = 7$ defines plane
quadric: ellipsoid,paraboloid,hyperboloid,cylinder,cone

Distance formulas

distance $d(P, Q) = |PQ| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$
distance point-plane: $d(P, \Sigma) = |(PQ) \cdot n|/|n|$
distance point-line: $d(P, L) = |(PQ) \times u|/|u|$
distance line-line: $d(L, M) = |(PQ) \cdot (u \times v)|/|u \times v|$
distance parallel lines L, M: distance point $d(P, M)$ where P is in L.
distance parallel planes: $d(P, \Sigma)$ where P is in first plane.

Functions

graph: $z = f(x, y)$
contour curve: $f(x, y) = c$ is a curve in the plane
contour map: draw curves $f(x, y) = c$ for various c.
contour surface: $f(x, y, z) = c$ in space

Curves

plane and space curves $r(t)$
circle: $x^2 + y^2 = r^2$, $r(t) = [r \cos t, r \sin t]^T$.
ellipse: $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$, $r(t) = [x_0 + a \cos t, y_0 + b \sin t]^T$
velocity $r\,'(t)$, acceleration $r\,''(t)$, $|r\,'(t)|$ speed
unit tangent vector $T(t) = r\,'(t)/|r\,'(t)|$
integration: get $r(t)$ from $r\,'(t)$ and $r(0)$ by integration.
integration: get $r(t)$ from acceleration $r\,''(t)$ as well as $r\,'(0)$ and $r(0)$.
$r\,'(t)$ is tangent to the curve at the point $r(t)$.
$r(t) = [f(t) \cos(t), f(t) \sin(t)]^T$ polar curve to polar graph $r = f(\theta)$.
$\int_a^b |r\,'(t)|\,dt$, arc length of parametrized curve.
$N(t) = T'(t)/|T'(t)|$ normal vector, is perpendicular to $T(t)$.
$B(t) = T(t) \times N(t)$ bi-normal vector, is perpendicular to T and N.
$\kappa(t) = |T\,'(t)|/|r\,'(t)|$ curvature $= |r\,'(t) \times r\,''(t)|/|r\,'(t)|^3$.
$\kappa(t)$ and arc length are independent of parametrization

Coordinates

- Cartesian coordinates (x, y, z)
- polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$, $r \geq 0$
- cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$, $r \geq 0$
- spherical coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$
- radius: $r = x^2 + y^2$ and spherical radius $\rho = x^2 + y^2 + z^2$.
- radius: important relation $r = \rho\sin(\phi)$
- Jacobian matrix
- Distortion factor

Surfaces

- $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
- $r = f(z, \theta)$ cylindrical surface, $r = r(z)$ surface of revolution
- $g(\rho, \theta, \phi) = 0$ spherical surface: example $\rho = 1$ sphere
- $f(x, y) = c$ level curves of $f(x, y)$
- plane: $ax + by + cz = d$, $r(s, t) = r_0 + sv + tw$, $[a, b, c]^T = v \times w$
- surface of revolution: $x^2 + y^2 = r(z)^2$, $r(\theta, z) = [r(z)\cos(\theta), r(z)\sin(\theta), z]^T$
- graph: $g(x, y, z) = z - f(x, y) = 0$, $r(x, y) = [x, y, f(x, y)]^T$
- rotated graph $g(x, y, z) = y - f(x, z) = 0$, $r(x, z) = [x, f(x, z), z]^T$
- ellipsoid: $r(\theta, \phi) = [a\cos\theta\sin\phi, b\sin\theta\sin\phi, c\cos\phi]^T$
- unit sphere: $x^2 + y^2 + z^2 = 1$, $r(u, v) = [\cos u\sin v, \sin u\sin v, \cos v]^T$

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