Unit 10: Coordinates

Introduction

10.1. Algebra is a powerful tool in geometry. In this lecture we circle back to the concept of coordinates and look also at other coordinate systems. We have introduced space as column vectors like $[1,2,3]^T$. We can think of it as an arrow from the origin to the point $(1,2,3)$. One speaks of the numbers appearing in $(1,2,3)$ as coordinates while the entries in $[1,2,3]^T$ are components of the vector. Most of the time we do not distinguish between the point $(1,2,3)$ and the vector $[1,2,3]^T$ as the two objects can clearly be identified naturally. We look in this lecture also at other coordinates like Polar and spherical coordinates. This will be important when doing integration.

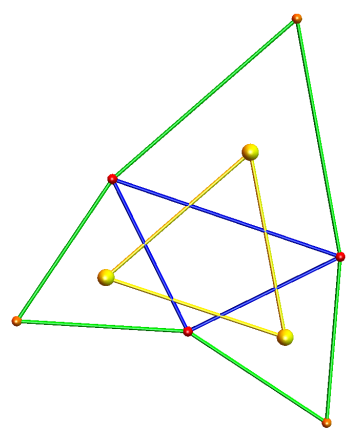

Figure 1. Napoleon's theorem tells that if we draw equilateral triangles over the sides of a triangle, their center of mass are on an equilateral triangle. A geometric proof is not so easy to find but using coordinates it is a direct calculation: for three complex numbers a, b, c, then $u = (a+b)/2+i(b-a)/3, v = (b+c)/2+i(c-b)/2, w = (a+b)/2+i(b-a)/3$ satisfy $|u-v| = |u-v| = |v-w|$. The result is famous because no other theorem has been rediscovered so many times. While Napoleon might never have discovered or proved it himself, he kept conversations with mathematicians like Lagrange or Fourier.

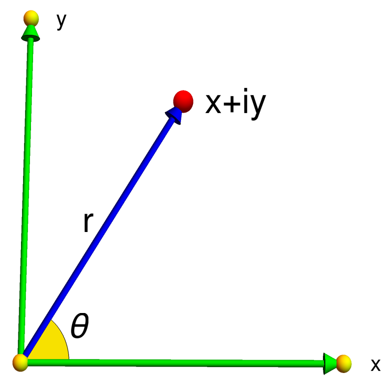

FIGURE 2. In the two-dimensional plane a point $(x, y) = (3, 4)$ can also be identified with the complex number $z = x + iy = 3 + 4i$ or the vector $[3, 4]^T$. The magnitude of the vector is $\sqrt{x^2 + y^2}$ and is defined to be the length of the complex number z. Multiplication rotates and scales. A multiplication with i rotates by 90 degrees.

LECTURE

10.2. It was René Descartes who in 1637 introduced **coordinates** and brought algebra close to geometry. [1] The **Cartesian coordinates** (x, y) in $\mathbb{R}^2$ can be replaced by other coordinate systems like **polar coordinates** (r, θ), where $r = \sqrt{x^2 + y^2} \geq 0$ is the **radial distance** to the $(0, 0)$ and $\theta \in [0, 2\pi)$ is the **polar angle** made with the positive x-axis. Since θ is in the interval $[0, 2\pi)$, it is best described in the complex notation $\theta = \arg(x + iy)$. The radius $r = |z| = \sqrt{x^2 + y^2}$ is the length of the complex number. The conversion from the (r, θ) coordinates to the (x, y)-coordinates is

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

The radius is $r = \sqrt{x^2 + y^2}$, where if non-zero, we always take the positive root. The angle formula $\arctan(y/x)$ only holds if x and y are both positive. The angle θ is not uniquely defined at the origin $(0, 0)$, most software just assumes $\arg(0) = 0$.

10.3. We can write a vector in $\mathbb{R}^2$ also in the form of a **complex number** $z = x + iy \in \mathbb{C}$ with symbol i. This is not only notational convenience. Complex numbers can be added and multiplied like other numbers and while $\mathbb{R}^2 = \mathbb{C}$, the later has a **multiplicative structure**. In order to fix that structure, one only needs to specify that $i^2 = -1$. This gives $(a + ib)(c + id) = ac - bd + i(ad + bc)$. We also have $|a + ib| = \sqrt{(a + ib)(a - ib)} = \sqrt{a^2 + b^2}$. An important **formula of Euler** link the exponential and trigonometric functions:

Theorem: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

[1] Descartes: La Géometrie, 1637 (1 year after the foundation of Harvard college)

10.4. The proof is to write the series definition on both sides. First recall the definitions of $e^x = 1+x+x^2/2!+x^3/3!+....$ If we plug in $x = i\theta$ we get $e^{i\theta} = 1+i\theta-\theta^2/2!-i\theta^3/3!+\theta^4/4!...$ But this is $(1-\theta^2/2+\theta^4/4!...) + i(\theta-\theta^3/3!+\theta^5/5!-...)$ which is $\cos(\theta)+i\sin(\theta)$. QED. If you prefer not to see the functions exp, sin, cos being **defined** as series, you can see them as **Taylor series** $f(x) = f(0)+f'(0)x+f''(0)/2!x^2+... = \sum_{k=0}^{\infty}(f^{(k)}(0)/k!)x^k$. By differentiating the functions at 0, we see then the connection.

10.5. The Euler formula implies for $\theta = \pi$ the magical formula

Theorem: $e^{i\pi}+1=0$

This formula is often voted the **"nicest formula in math"**. [2] It combines "analysis" in the form e, "geometry" in the form of π, "algebra" in the form of i, the additive unit 0 and the multiplicative unit 1.

10.6. The Euler formula allows to write any complex number as $z = re^{i\theta}$. Given an other complex number $w = se^{i\phi}$ we have $zw = rse^{i\theta+\phi}$ showing that the polar angles add and the radius multiplies. The Euler formula also allows to define the **logarithm** of any complex number as $\log(z) = \log(|z|) + i\arg(z) = \log(r) + i\theta$. We see now that going from (x, y) to $(\log(r), \theta)$ is a very natural transformation from $\mathbb{C} \setminus 0$ to $\mathbb{C}$. The exponential function $\exp : z \to e^z$ is a map from $\mathbb{C} \to \mathbb{C} \setminus 0$. It transforms the additive structure on $\mathbb{C}$ to the multiplicative structure because $\exp(z+w) = \exp(z)\exp(w)$.

10.7. In three dimensions, we can look at **cylindrical coordinates** (r, θ, z) which are just polar coordinates in the first two coordinates. A cylinder of radius 2 for example is given as $r = 2$. The torus $(3+x^2+y^2+z^2)^2 - 16(x^2+y^2) = 0$ can be written as $3+r^2+z^2 = 4r$ or more intuitively as $(r-2)^2+z^2 = 1$, a circle in the $r-z$ plane.

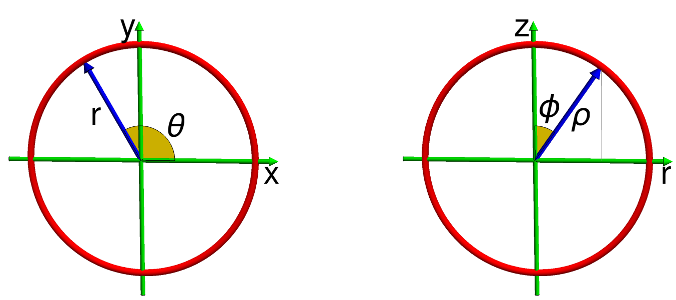

FIGURE 3. Key pictures to derive cylindrical and spherical coordinates.

10.8. The **spherical coordinates** (ρ, θ, ϕ), where $\rho = \sqrt{x^2+y^2+z^2}$. The angle θ is the polar angle as in cylindrical coordinates and ϕ is the angle between the point (x, y, z) and the z-axis. We have $\cos(\phi) = [x, y, z]\cdot[0, 0, 1]/|[x, y, z]| = z/\rho$ and $\sin(\phi) = |[x, y, z]\times[0, 0, 1]|/|[x, y, z]| = r/\rho$ so that $z = \rho\cos(\phi)$ and $r = \rho\sin(\phi)$ and therefore

[2] D. Wells, Which is the most beautiful?, Mathematical Intelligencer, 1988

$$\begin{array}{rcl} x &=& \rho\sin(\phi)\cos(\theta) \\ y &=& \rho\sin(\phi)\sin(\theta) \\ z &=& \rho\cos(\phi) \end{array}$$

where $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$ and $\rho \geq 0$.

10.9. A **coordinate change** $x \to f(x)$ in the plane is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. A point (x_1, x_2) is mapped into (f_1, f_2). We write ∂_{x_k} for the **partial derivative** with respect to the variable x_k. For example $\partial_{x_1}(x_1^2 x_2 + 3x_1 x_2^3) = 2x_1 x_2 + 3x_2^3$.

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}, \quad df\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) \end{bmatrix},$$

where df is a matrix called the **Jacobian matrix**. The determinant is called the **distortion factor** at $x = (x_1, x_2)$.

10.10. For polar coordinates, we get

$$f\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}, df\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}.$$

Its distortion factor of the Polar map is r. We will use this when integrating in polar coordinates.

10.11. If $f(z) = z^2 + c$ with $c = a + ib, z = x + iy$ is written as $f(x, y) = (x^2 - y^2 + a, 2xy + b)$, then df is a 2×2 **rotation dilation matrix** which corresponds to the complex number $f'(z) = 2z$. The algebra $\mathbb{C}$ is the same as the algebra of rotation-dilation matrices.

10.12. A **coordinate change** $x \to f(x)$ in space is a map $f : \mathbb{R}^3 \to \mathbb{R}^3$. We compute

$$f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}, df\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) & \partial_{x_3} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) & \partial_{x_3} f_2(x) \\ \partial_{x_1} f_3(x) & \partial_{x_2} f_3(x) & \partial_{x_3} f_3(x) \end{bmatrix}.$$

We wrote $x = (x_1, x_2, x_3)$. Its determinant $\det(dT)(x)$ is a volume distortion factor.

10.13. For spherical coordinates, we have

$$f\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{bmatrix}, df\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{bmatrix}.$$

The distortion factor is $\det(df(\rho, \phi, \theta)) = \rho^2 \sin(\phi)$.

Examples

10.14. The point $(x, y) = (-1, 1)$ corresponds to the complex number $z = -1 + i$. It has the polar coordinates $(r, \theta) = (\sqrt{2}, 3\pi/4)$. As we have $z = re^{i\theta}$, we check $z^2 = (-1 + i)(-1 + i) = -2i$ which agrees with $(re^{i\theta})^2 = r^2 e^{2i\theta} = 2e^{6\pi i/4}$.

10.15. a) $(x, y, z) = (1, 1, -\sqrt{2})$ corresponds to spherical coordinates $(\rho, \phi, \theta) = (2, 3\pi/4, \pi/4)$.
b) The point given in spherical coordinates as $(\rho, \phi, \theta) = (3, 0, \pi/2)$ is the point $(0, 3, 0)$.

10.16. a) The set of points with $r = 1$ in $\mathbb{R}^2$ form a circle.
b) The set of points with $\rho = 1$ in $\mathbb{R}^3$ form a sphere.
c) The set of points with spherical coordinates $\phi = 0$ are points on the positive z-axis.
d)The set of points with spherical coordinates $\theta = 0$ form a half plane in the yz-plane.
e) The set of points with $\rho = \cos(\phi)$ form a sphere. Indeed, by multiplying both sides with ρ, we get $\rho^2 = \rho\cos(\phi)$ which means $x^2 + y^2 + z^2 = z$, which is after a completion of the square equal to $x^2 + y^2 + (z - 1/2)^2 = 1/4$.

10.17. For $A \in M(n,n)$, $f(x) = Ax + b$ has $df = A$ and distortion factor $\det(A)$.

10.18. Find the Jacobian matrix and distortion factor of the map $f(x_1, x_2) = (x_1^3 + x_2, x_2^2 - \sin(x_1))$. Answer: Write both the transformation and the Jacobian:

$$f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^3 + x_2 \\ x_2^2 - \sin(x_1) \end{bmatrix}, \; df\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1^2 & 1 \\ -\cos(x_1) & 2x_2 \end{bmatrix}.$$

The Jacobian matrix is $\det(df(x)) = 6x_1^2x_2 + \cos(x_1)$.

Illustrations

10.19. Let $T : \mathbb{C} \to \mathbb{C}$ be defined as $z \to z^2 + c$, where $z = x + iy$. The set of all $c = a + ib$ for which the iterates $T^n(0)$ stay bounded is the **Mandelbrot set** M. For $c = -1$ we get $T(0) = -1, T^2(0) = T(-1) = 0$ so that $T^n(z)$ is either 0 or -1. The point $c = -1$ is in M. The point $c = 1$ gives $T(0) = 1, T^2(0) = 1^2 = 1 = 2, T^3(0) = 2^2 + 1 = 5$. Induction shows that $T^n(0)$ does not converge. The point $c = 1$ is not in M.

10.20. If T is the transformation in $\mathbb{R}^3$ which is in spherical coordinates given by $T(x) = x^2 + c$, where x^2 has spherical coordinates $(\rho^2, 2\phi, 2\theta)$ if x has (ρ, ϕ, θ). It turns out that $T(x) = x^8 + c$ gives a nice analogue of the Mandelbrot set, the **Mandelbulb**.

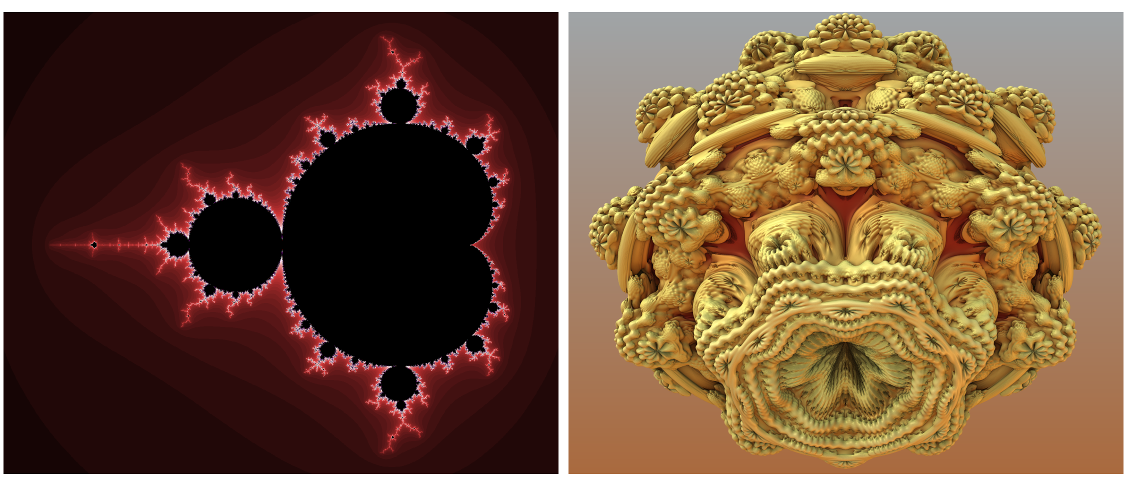

FIGURE 4. The **Mandelbrot set** $M = \{c \in \mathbb{C} | T(z) = z^2 + c$ has bounded $T^n(0)\}$. There is a similar construction in space $\mathbb{R}^3$ which uses spherical coordinates. This leads to the **Mandelbulb set** $B = \{c \in \mathbb{R}^3 \mid T(x) = x^8 + c \text{ has bounded } T^n(0)\}$, where x^8 has spherical coordinates $(\rho^8, 8\phi, 8\theta)$ if x has spherical coordinates (ρ, ϕ, θ).

Homework

Problem 10.1: a) Find the polar coordinates of $(x, y) = (-1, \sqrt{3})$.
b) Which point has the polar coordinates $(r, \theta) = (2, \pi/4)$?
c) Find the spherical coordinates of the point $(x, y, z) = (1, 1, \sqrt{2})$.
d) Which point has the spherical coordinates $(\rho, \theta, \phi) = (3, \pi/2, \pi/3)$?

Problem 10.2: a) Compute $T_c^n(0)$ for $c = (1 + i)$ for $n = 1, 2, 3$. Is $1 + i$ in the Mandelbrot set?
b) What is the "eye for an eye" number i^i? (You can use $z^w = e^{w \log(z)}$).

Problem 10.3: a) Which surface is described as $r = z$?
b) Describe the hyperbola $x^2 - y^2 = 5$ in polar coordinates.
c) Which surface is described as $\rho \sin(\phi) = \rho^2$?
d) Describe the hyperboloid $x^2 + y^2 - z^2 = 1$ in spherical coordinates.

Problem 10.4: a) Compute the Jacobian matrix and distortion factor of the coordinate change $T(x, y) = (2x + \sin(x) - y, x)$ (**Chirikov map**).
b) Compute the Jacobian matrix and distortion factor of the coordinate change $T(x, y) = (1 - 1.4x^2 - y, 0.3x)$. (Classical **Hénon map**).
P.S. When you do the coordinate change of the Chiriov map again and again, one can observe **chaos**. In the case of the Hénon map, one sees a **strange attractor**, a fractal object which similarly as the Koch curve encountered last week has a dimension larger than 1.

Problem 10.5: a) Verify that the Mandelbrot set M is contained in the set $|c| \leq 2$. As a reminder, this means you have to show that then $0 \to c \to c^2 + c \to (c^2 + c)^2 + c...$ escapes to infinity.
b) Optional: Use the same argument to see that the Mandelbulb set B is contained in the set $|c| \leq 2$.

Oliver Knill, knill@math.harvard.edu, Math 22b, Harvard College, Spring 2022