LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 11: Parametrization

Introduction

11.1. We have seen that when parametrizing curves $r(t)$, we have much more control than when looking at curves given by equations. It would be difficult to describe a helix $r(t) = [\cos(t), \sin(t), t]$ in terms of equations for example. For surfaces also, it is good to have as many coordinates as the dimension. We live on a two dimensional sphere $x^2 + y^2 + z^2 = 1$ but do not use the x, y, z coordinates to describe a point on the surface. We use two coordinates longitude) and latitude Euler used first the parametrization $[x, y, z] = [\cos(t)\cos(s), \sin(t)\sin(s), \sin(s)]$ where t, s are angles. You can check quickly that $x^2 + y^2 + z^2$ adds up to 1 so that whatever angles t, s we chose, we always are on the sphere.

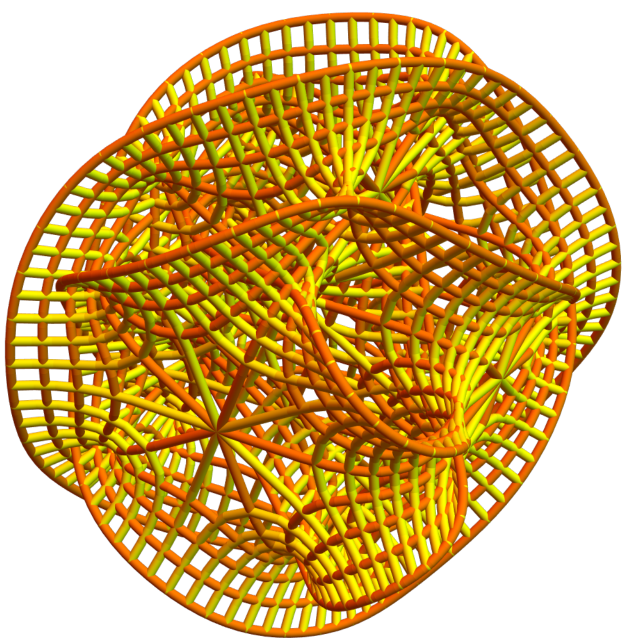

FIGURE 1. This surface is an example of a Calabi-Yau surface. It is parametrized $r(u, v)$. We drew out some grid curves, where u is constant or v is constant.

Lecture

11.2. A map $r : \mathbb{R}^m \to \mathbb{R}^n$ is called a **parametrization**. We have seen maps r from $\mathbb{R}$ to $\mathbb{R}^n$, which were **curves**. Then we have seen maps $f : \mathbb{R}^n \to \mathbb{R}^n$ which were **coordinate changes**. In each case we defined the **Jacobian matrix** $df(x)$. In the case of the curve $r : \mathbb{R} \to \mathbb{R}^n$, it was the **velocity** $dr(t) = r'(t)$. In the case of coordinate changes, the Jacobian matrix $df(x)$ was used to get the **volume distortion factor** $\det(df(x)) = \sqrt{\det(df^T df)}$. Today, we look at the case $m < n$. In particular at $m = 2, n = 3$. As in the case of curves, we use the letter r to describe the map. The image of a map $r : R \subset \mathbb{R}^m \to \mathbb{R}^n$ is then a **m-dimensional surface** in $\mathbb{R}^n$. The **distortion factor** $||dr||$ defined as $||dr||^2 = \det(dr^T dr)$ will be used later to compute **surface area**. [1]

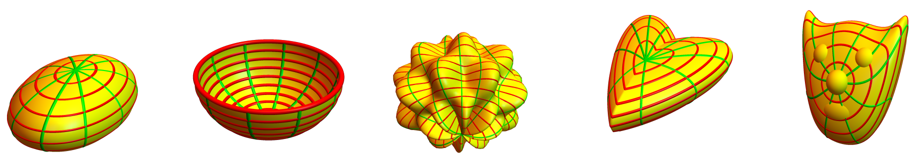

Figure 2. An ellipsoid, half an ellipsoid, a bulb, a heart and a cat.

11.3. We mostly discuss here the case $m = 2$ and $n = 3$, as we ourselves are made of two-dimensional surfaces, like cells, membranes, skin or tissue. A map $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$, written as $r(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ defines a two-dimensional surface. In order to save space, we also just write $r(u, v) = [x(u, v), y(u, v), z(u, v)]$. In computer graphics, the r is called uv-**map**. The uv-plane is where you draw a texture. The map r places it onto the surface. In geography, the map r is called (surprise!) a **map**. Several maps define an **atlas**. The curves $u \to r(u, v)$ and $v \to r(u, v)$ are called **grid curves**.

11.4. The parametrization $r(\phi, \theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ produces the **sphere** $x^2 + y^2 + z^2 = 1$. The full sphere has $0 \le \phi \le \pi$, $0 \le \theta < 2\pi$. By modifying the coordinates, we get an **ellipsoid** $r(\phi, \theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$ satisfying $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By allowing a, b, c to be functions of ϕ, θ we get "bumpy spheres" like $r(\phi, \theta) = (3 + \cos(3\phi)\sin(4\theta))[\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$.

11.5. **Planes** are described by linear maps $r(x) = Ax + b$ with $A \in M(3, 2)$ and $b \in M(3, 1)$. The Jacobian map is $dr = A$. Let r_u, r_v be the two column vectors of A. Actually, r_u is a short cut for $\partial_u r(u, v)$, which is the velocity vector of the **grid curve** $u \to r(u, v)$.

[1] Distinguish $||A||^2 = \det(A^T A)$ and $|A|^2 = \mathrm{tr}(A^T A)$ in $M(n, m)$. They only agree for $m = 1$.

11.6. An example is the parametrization $r(u,v) = [u+v-1, u-v+3, 3u-5v+7]$. In this case $b = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}$, $r_u = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ $r_v = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}$ and $A = dr = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & -5 \end{bmatrix}$. We see $A^T A = \begin{bmatrix} 11 & -15 \\ -15 & 27 \end{bmatrix}$ which has determinant 72. We also have

$$|r_u \times r_v|^2 = |\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}|^2 = |\begin{bmatrix} -2 \\ 8 \\ -2 \end{bmatrix}|^2 = 72$$

11.7. The previous computation suggests a relation between the normal vector and the fundamental form $g = dr^T dr$. In three dimensions, the distortion factor of a parametrization $r : \mathbb{R}^2 \to \mathbb{R}^3$ can indeed always be rewritten using the cross product:

Theorem: $\det(dr^T dr) = |r_u \times r_v|^2$.

Proof. As $dr^T dr = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$, the identity is the **Cauchy-Binet identity** $|r_u \times r_v|^2 = |r_u|^2 |r_v|^2 - |r_u \cdot r_v|^2$ which boils down to $\sin^2(\theta) = 1 - \cos^2(\theta)$, where θ is the angle between r_u and r_v. This is the angle between the grid curves you see on the pictures.

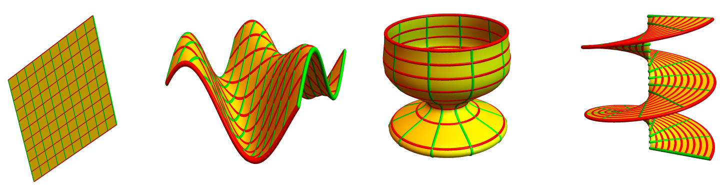

FIGURE 3. A plane, graph, surface of revolution and helicoid.

EXAMPLES

11.8. For the **unit sphere** $r(\phi,\theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ and $A = dr$:

$$g = A^T A = \begin{bmatrix} \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \\ \cos(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) \\ -\sin(\phi) & 0 \end{bmatrix}$$

This is $g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2(\phi) \end{bmatrix}$ and $\sqrt{\det(g)} = \sin(\phi)$ is the distortion factor.

11.9. An important class of surfaces are **graphs** $z = f(x,y)$. Its most natural parametrization is $r(x,y) = [x, y, f(x,y)]$, where the map r just lifts up the bottom part to the elevated version. An example is the elliptic paraboloid $r(x,y) = [x, y, x^2 + y^2]$ and the hyperbolic paraboloid $r(x,y) = [x, y, x^2 - y^2]$. We could of course have written also $r(u,v) = [u, v, u^2 - v^2]$.

11.10. A **surface of revolution** is parametrized like $r(\theta, z) = [g(z)\cos(\theta), g(z)\sin(\theta), z]$. Note that we can use any variables. In this case, $u = \theta, v = z$ are used. An example is the **cone** $r(\theta, z) = [z\cos(\theta), z\sin(\theta), z]$ or the **one-sheeted hyperboloid** $r(\theta, z) = [\sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta), z]$.

11.11. The **torus** is in cylindrical coordinates given as $(r-3)^2 + z^2 = 1$. We can parametrize this using the polar angle θ and the polar angle centered at center of the circle as $r(\theta, \phi) = [(3+\cos(\phi))\cos(\theta), (3+\cos(\phi))\sin(\theta), \sin(\phi)]$. Both angles θ and ϕ go from 0 to 2π. We see now also the relation with the **toral coordinates**.

11.12. The **helicoid** is the surface you see as a staircase or screw. The parametrization is $r(\theta, p) = [p\cos(\theta), p\sin(\theta), \theta]$. How can we understand this? The key is to look at grid curves. If $p = 1$, we get a curve $r(\theta) = [\cos(\theta), \sin(\theta), \theta]$ which we had identified as a **helix**. On the other hand, if you fix θ, then you get lines.

11.13. Side remark. The **first fundamental form** $g = dr^T dr$ is also called a **metric tensor**. In **Riemannian geometry** one looks at a manifold M equipped with a metric g. The simplest case is when g comes from a parametrization, as we did here. In physics, we know that it is **mass** which deforms space-time. The quantity $||g||^2 = \det(g)$ is a multiplicative analogue of $|g|^2 = \text{tr}(g)$. For an invertible positive definite square matrix A, we will later see the identity $\log\det(A) = \text{tr}\log(A)$ which illustrates how both determinant and trace are pivotal numerical quantities derived from a matrix. Trace is **additive** because of $\text{tr}(A+B) = \text{tr}(A)+\text{tr}(B)$ and determinant is **multiplicative** $\det(AB) = \det(A)\det(B)$ as we will see later.

11.14. To summarize, we have seen so far that there are two fundamentally different ways to describe a manifold. The first is to write it as a level surface $f = c$ which is a **kernel** of a map $g(x) = f - c$. A second is to write it as the **image** of some map r.

Illustration

Figure 4. "Veritas on Earth and the Moon" theme (rendered in Povray).

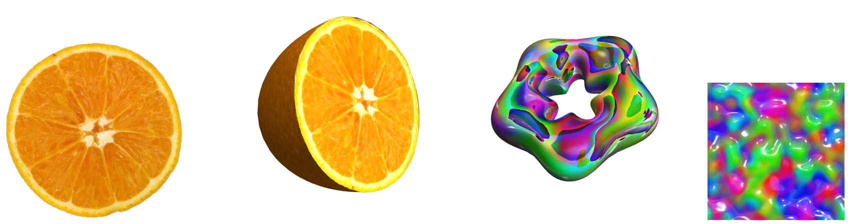

Figure 5. A fruit and math-candy© math-candy.com (rendered in Mathematica)

Homework

Problem 11.1: Parametrize the upper part of the two sheeted hyperboloid $x^2 + y^2 - z^2 = -1, z > 0$ as a surface of revolution.

Problem 11.2: a) Parametrize the plane $x + 2y + 3z - 6 = 0$ using a map $r : \mathbb{R}^2 \to \mathbb{R}^3$. b) Now find the matrix $A = dr$ and compute $g = A^T A$ as well as the distortion factor $\sqrt{\det(A^T A)}$. c) Also compute r_u, r_v and $r_u \times r_v$ and then compute $|r_u \times r_v|$. You should get the same number.

Problem 11.3: Given a parametrization $r(\theta, \phi) = [(7 + 2\cos(\phi))\cos(\theta), (7 + 2\cos(\phi))\sin(\theta), 2\sin(\phi)]$ of the 2-torus, find the implicit equation $g(x, y, z) = 0$ which describes this torus.

Problem 11.4: Parametrize the hyperbolic paraboloid $z = x^2 - y^2$. What is the first fundamental form $g = dr^T dr$ which is $g = \begin{bmatrix} r_x \cdot r_x & r_x \cdot r_y \\ r_y \cdot r_x & r_y \cdot r_y \end{bmatrix}$?. What is the distortion factor $\sqrt{\det(g)}$?

Problem 11.5: The matrix $g = dr^T dr$ is also called the **first fundamental form**. If $r : \mathbb{R}^4$ to $\mathbb{R}^4$ is a parametrization of **space time** then g is the **space time metric tensor**. The matrix entries of g appear in **general relativity**. Now for some reasons, physics folks use Greek symbols to access matrix entries. They write $g_{\mu\nu}$ for the entry at row μ and column ν. This appears for example in the **Einstein field equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \ .$$

We just want you to look up the equation and tell from each of the variables, what it is called and whether it is a matrix, a scalar function or a constant.