Unit 13: Partial differential equations

Introduction

13.1. If we can relate the changes in one quantity with changes in an other quantity, **partial differential equations** come in. One of the simplest rules is that the rate of change of a function $f(t,x)$ in time is related to the rate of change in space. Such a rule could be expressed for example as a rule $f_t(t,x) = f_x(t,x)$, where f_t is the partial derivative with respect to t and f_x is the partial derivative with respect to x. You can check that $f(t,x) = \sin(t+x)$ is an example of a function which satisfies this differential equation. You can see even that for any function g, the function $f(t,x) = g(t+x)$ satisfies $f_t = f_x$. A typical situation is to be given $f(0,x)$, the situation of "now". We then can see what $f(t,x)$ is for a **later time** t. This describes the situation in the future. As you see, the differential equation $f_t = f_x$ describes "transport". The initial situation is translated to the left. Check this out and draw for example $f(0,x) = x^2$. We see that $f(t,x) = (x+t)^2$ and especially $f(1,x) = (x+1)^2$. The graph has moved to the left.

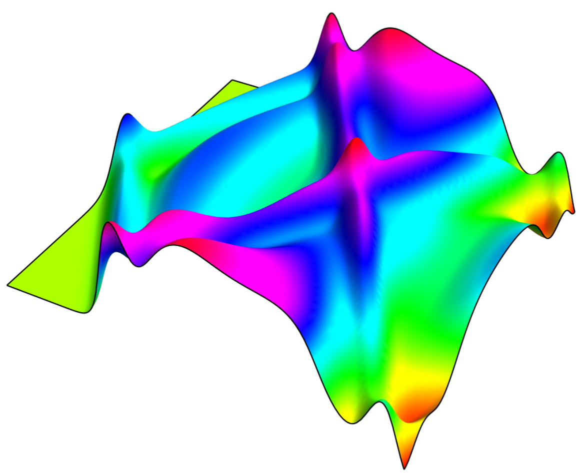

FIGURE 1. A function $f(t,x)$ satisfying a differential equation $f_{tt} - f_{xx} = \sin(u)$. This PDE is called Sin $-$ *Gordon* equation, a nonlinear wave equation featuring **solitons**. Space is here one dimensional time goes from left to right. We see a wave going left and right, reflecting at the boundary and building up to a larger peak. A "rogue wave".

Lecture

13.2. A **partial differential equation** is a rule which combines the rates of changes of different variables. Our lives are affected by partial differential equations: the **Maxwell equations** describe electric and magnetic fields E and B. Their motion leads to the propagation of light. The **Einstein field equations** relate the metric tensor g with the mass tensor T. The **Schrödinger equation** tells how quantum particles move. Laws like the **Navier-Stokes equations** govern the motion of fluids and gases and especially the currents in the ocean or the winds in the atmosphere. Partial differential equations appear also in unexpected places like in finance, where for example, the **Black-Scholes equation** relates the prices of options in dependence of time and stock prices.

13.3. If $f(x,y)$ is a function of two variables, we can differentiate f with respect to both x or y. We just write $f_x(x,y)$ for $\partial_x f(x,y)$. For example, for $f(x,y) = x^3y + y^2$, we have $f_x(x,y) = 3x^2y$ and $f_y(x,y) = x^3 + 2y$. If we first differentiate with respect to x and then with respect to y, we write $f_{xy}(x,y)$. If we differentiate twice with respect to y, we write $f_{yy}(x,y)$. An equation for an unknown function f for which partial derivatives with respect to at least two different variables appear is called a **partial differential equation** PDE. If only the derivative with respect to one variable appears, one speaks of an **ordinary differential equation** ODE. An example of a PDE is $f_x^2 + f_y^2 = f_{xx} + f_{yy}$, an example of an ODE is $f'' = f^2 - f'$. It is important to realize that it is a function we are looking for, not a number. The ordinary differential equation $f' = 3f$ for example is solved by the functions $f(t) = Ce^{3t}$. If we prescribe an initial value like $f(0) = 7$, then there is a unique solution $f(t) = 7e^{3t}$. The **KdV partial differential equation** $f_t + 6ff_x + f_{xxx} = 0$ is solved by (you guessed it) $2\text{sech}^2(x-4t)$. This is one of many solutions. In that case they are called **solitons**, nonlinear waves. Korteweg-de Vries (KdV) is an icon in a mathematical field called **integrable systems** which leads to insight in ongoing research like about **rogue waves** in the ocean.

13.4. We say $f \in C^1(\mathbb{R}^2)$ if both f_x and f_y are continuous functions of two variables and $f \in C^2(\mathbb{R}^2)$ if all f_{xx}, f_{yy}, f_{xy} and f_{yx} are continuous functions. The next theorem is called the **Clairaut theorem**. It deals with the partial differential equation $f_{xy} = f_{yx}$. The proof demonstrates the **proof by contradiction**. We will look at this technique a bit more in the proof seminar.

Theorem: Every $f \in C^2$ solves the Clairaut equation $f_{xy} = f_{yx}$.

13.5. Proof. We use **Fubini's theorem** which will appear later in the double integral lecture: integrate $\int_{x_0}^{x_0+h}(\int_{y_0}^{y_0+h} f_{xy}(x,y)\,dy)dx$ by applying the **fundamental theorem of calculus** twice $\int_{x_0}^{x_0+h} f_x(x,y_0+h) - f_x(x,y_0)\,dx = f(x_0+h,y_0+h) - f(x_0,y_0+h) - f(x_0+h,y_0) + f(x_0,y_0)$. An analogous computation gives $\int_{y_0}^{y_0+h}(\int_{x_0}^{x_0+h} f_{yx}(x,y)\,dx)dy = f(x_0+h,y_0+h) - f(x_0,y_0+h) - f(x_0+h,y_0) + f(x_0,y_0)$. Fubini applied to $g(x,y) = f_{xy}(x,y)$ assures $\int_{y_0}^{y_0+h}(\int_{x_0}^{x_0+h} f_{yx}(x,y)\,dx)dy = \int_{x_0}^{x_0+h}(\int_{y_0}^{y_0+h} f_{yx}(x,y)\,dy)dx$ so that $\int\int_A f_{xy} - f_{yx}\,dydx = 0$. $\boxed{\text{Assume}}$ there is some (x_0,y_0), where $F(x_0,y_0) = f_{xy}(x_0,y_0) - f_{yx}(x_0,y_0) = c > 0$, then also for small h, the function F is bigger than $c/2$ everywhere on $A = [x_0,x_0+h] \times [y_0,y_0+h]$ so that $\int\int_A F(x,y)\,dxdy \geq \text{area}(A)c/2 = h^2c/2 > 0$ $\boxed{\text{contradicting}}$ that the integral is zero.

13.6. The statement is false for functions which are only C^1. The standard counter example is $f(x,y) = 4xy(y^2 - x^2)/(x^2 + y^2)$ which has for $y \neq 0$ the property that $f_x(0,y) = 4y$ and for $x \neq 0$ has the property that $f_y(x,0) = -4x$. You can see the comparison of $f(x,y) = 2xy = r^2\sin(2\theta)$ and $f(x,y) = 4xy(y^2 - x^2)/(x^2 + y^2) = r^2\sin(4\theta)$. The later function is not in C^2. The values f_{xy} and f_{yx}, changes of slopes of tangent lines, turn differently.

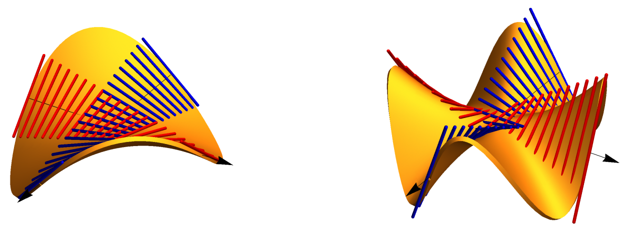

FIGURE 2. Clairaut holds for $f(x,y) = 2xy$ which is in polar coordinates $r^2\sin(2\theta)$. It does not for the function $f(x,y) = 4xy(y^2-x^2)/(x^2+y^2)$ which is in polar coordinates $2r^2\sin(2\theta)\cos(2\theta) = r^2\sin(4\theta)$.

ILLUSTRATION

13.7. In many cases, one of the variables is **time** for which we use the letter t and keep x as the **space variable**. The differential equation $f_t(t,x) = f_x(t,x)$ is called the **transport equation**. What are the solutions if $f(0,x) = g(x)$? Here is a cool derivation: if $Df = f'$ is the derivative, [1] we can build operators like $(D+D^2+4D^4)f = f'+f''+4f''''$. The transport equation is now $f_t = Df$. Now as you know from calculus, the only solution of $f' = af, f(0) = b$ is be^{at}. If we boldly replace the number a with with the operator D we get $f' = Df$ and get its solution

$$e^{Dt}g(x) = (1 + Dt + D^2t^2/2! + \cdots)g(x) = g(x) + g'(x)t + g''(x)t^2/2! + \cdots .$$

By the **Taylor formula**, this is equal to $g(x+t)$. You should actually remember Taylor as $\boxed{g(x+t) = e^{Dt}g(x)}$. We have derived for $g(x) = f(0,x)$ in $C^1(\mathbb{R}^2)$:

Theorem: $f_t = f_x$ is solved by $f(t,x) = g(x+t)$.

Proof. We can ignore the derivation and verify this very quickly: the function satisfies $f(0,x) = g(x)$ and $f_t(t,x) = f_x(t,x)$. QED.

13.8. Another example of a partial differential equation is the **wave equation** $f_{tt} = f_{xx}$. We can write this $(\partial_t + D)(\partial_t - D)f = 0$. One way to solve this is by looking at $(\partial_t - D)f = 0$. This means transport $f_t = f_x$ and $f(t,x) = f(x+t)$. We can also have $(\partial_t + D)f = 0$ which means $f_t = -f_x$ leading to $f(x-t)$. We see that every combination $af(x+t) + bf(x-t)$ with constants a,b is a solution. Fixing the constants a,b so that $f(x,0) = g(x)$ and $f_t(x,0) = h(x)$ gives the following **d'Alembert solution**. It requires $g,h \in C^2(\mathbb{R})$.

[1] We usually write df for derivative but D tells it is an operator. D also stands for Dirac.

Theorem: $f_{tt} = f_{xx}$ is solved by $f(t,x) = \frac{g(x+t)+g(x-t)}{2} + \frac{h(x+t)-h(x-t)}{2}$.

13.9. Proof. Just verify directly that this indeed is a solution and that $f(0,x) = g(x)$ and $f_t(0,x) = h(x)$. Intuitively, if we throw a stone into a narrow water way, then the waves move to both sides.

13.10. The partial differential equation $f_t = f_{xx}$ is called the **heat equation**. Its solution involves the **normal distribution**

$$N(m,s)(x) = e^{-(x-m)^2/(2s^2)}/\sqrt{2\pi s^2}$$

in probability theory. The number m is the **average** and s is the **standard deviation**.

13.11. If the initial heat $g(x) = f(0,x)$ at time $t = 0$ is continuous and zero outside a bounded interval $[a,b]$, then

Theorem: $f_t = f_{xx}$ is solved by $f(t,x) = \int_a^b g(m)N(m,\sqrt{2t})(x)\, dm$.

Proof. For every fixed m, the function $N(m,\sqrt{2t})(x)$ solves the heat equation.

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f=PDF[NormalDistribution[m,Sqrt[2 t]],x]; Simplify[D[f,t]==D[f,{x, 2}]]
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Every Riemann sum approximation $g(x) = (1/n)\sum_{k=1}^n g(m_k)$ of g defines a function $f_n(t,x) = (1/n)\sum_{k=1}^n g(m_k)N(m_k,\sqrt{2t})(x)$ which solves the heat equation. So does $f(t,x) = \lim_{n\to\infty} f_n(t,x)$. To check $f(0,x) = g(x)$ which need $\int_{-\infty}^{\infty} N(m,s)(x)\, dx = 1$ and $\int_{-\infty}^{\infty} h(x)N(m,s)(x)\, dx \to h(m)$ for any continuous h and $s \to 0$, proven later.

13.12. For functions of three variables $f(x,y,z)$ one can look at the partial differential equation $\Delta f(x,y,z) = f_{xx} + f_{yy} + f_{zz} = 0$. It is called the **Laplace** equation and Δ is called the **Laplace operator**. The operator appears also in one of the most important partial differential equations, the **Schrödinger equation**

$$i\hbar f_t = Hf = -\frac{\hbar^2}{2m}\Delta f + V(x)f \ ,$$

where $\hbar = h/(2\pi)$ is a scaled **Planck constant** and $V(x)$ is the potential depending on the position x and m is the mass. For $i\hbar f_t = Pf$ with $P = -i\hbar D$, then the solution $f(x-t)$ is forward translation. The operator P is the **momentum operator** in quantum mechanics. The Taylor formula tells that P generates translation.

Homework

Problem 13.1: Verify that for any constant b, the function

$$f(x,t) = e^{-bt}\sin(x+t)$$

satisfies the driven transport equation

$$f_t(x,t) = f_x(x,t) - bf(x,t) \ .$$

This PDE is sometimes called the **advection equation** with damping b.

Problem 13.2: We have seen in class that $f(t,x) = e^{-x^2/(4t)}/\sqrt{4\pi t}$ solves the heat equation $f_t = f_{xx}$. Verify more generally that

$$e^{-x^2/(4at)}/\sqrt{4a\pi t}$$

solves the **heat equation**

$$f_t = af_{xx} \ .$$

Problem 13.3: The **Eiconal equation** $f_x^2 + f_y^2 = 1$ is used in optics. Let $f(x,y)$ be the distance to the circle $x^2+y^2 = 1$. Show that it satisfies the eiconal equation. Remark: the equation can be written rewritten as $||df||^2 = 1$, where $df = \nabla f = [f_x, f_y]$ is the gradient of f which is the Jacobian matrix for the map $f : \mathbb{R}^2 \to \mathbb{R}$.

Problem 13.4: The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is a version of the **Black-Scholes equation**. Here $f(x,t)$ is the price of a **call option** and x is the stock price and t is time. Find a function $f(x,t)$ solving it which depends both on x and t. Hint: look first for solutions $f(x,t) = g(t)$ or $f(x,t) = h(x)$ and then for functions of the form $f(x,t) = g(t) + h(x)$.

Problem 13.5: The partial differential equation

$$f_t + ff_x = f_{xx}$$

is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which is used to describe the weather. Verify that the function

$$f(t,x) = \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

is a solution of the Burgers equation. You better use technology.