LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 20: Constraints

Introduction

20.1. There is rarely a "free lunch". If we want to maximize a quantity, we often have to work with constraints. Obstacles might prevent us to change the parameters arbitrarily. The gradient can still be used as a guiding principle. While we can not achieve ∇f to be zero, we can look for points where the gradient is perpendicular to the constraint. This gives us an optimal point under the confinement. If you hike on a path in the mountains, you often reach a local maximum without being on top of the mountain. What happens at such points x is that $\nabla f(x)$ is perpendicular to the curve meaning that $\nabla f(x)$ is parallel to $\nabla g(x)$.

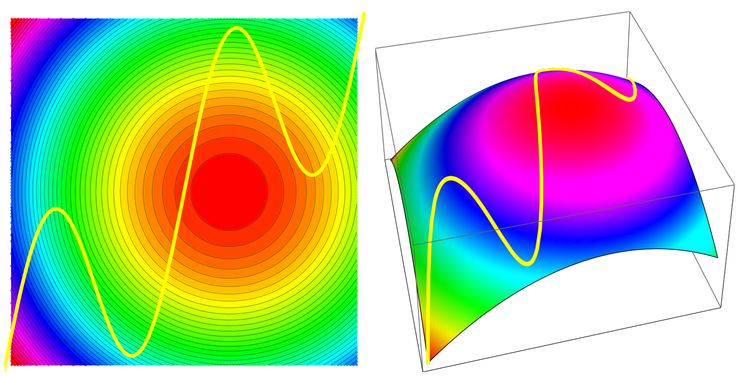


FIGURE 1. The situation, where a function $f(x, y)$ is optimized along a curve $g(x, y) = c$ is a frame-work which can be tackled with Lagrange. The condition of being maximal means that the gradient of f is perpendicular to the curve. This means that the gradients of f and g are parallel. $\nabla f = \lambda \nabla g$.

20.2. The method of Lagrange is much more general. We can work with arbitrary many constraints and still use the same principle. The gradient of $f : \mathbb{R}^n \to \mathbb{R}$ is then perpendicular to the constraint surface which means that is a linear combination of the gradients of all the m constraints: these are n equations $\nabla f = \sum_{j=1}^m \lambda_j \nabla g_j$ because the vectors have n components. Together with the m equations $g_j = c_j$ we have $n+m$ equations for $n+n$ variables $x_1, \dots, x_n, \lambda_1, \dots, \lambda_m$.

Lecture

20.3. If we want to maximize a function $f : \mathbb{R}^m \to \mathbb{R}$ on the constraint $S = \{x \in \mathbb{R}^m \mid g(x) = c\}$, then both the gradients of f and g matter. We call two vectors v, w **parallel** if $v = \lambda w$ or $w = \lambda v$ for some real λ. The zero vector is parallel to everything. Here is a variant of Fermat:

Theorem: If x_0 is a maximum of f under the constraint $g = c$, then $\nabla f(x_0)$ and $\nabla g(x_0)$ are parallel.

20.4. Proof by contradiction: assume $\nabla f(x_0)$ and $\nabla g(x_0)$ are not parallel and x_0 is a local maximum. Let T be the tangent plane to $S = \{g = c\}$ at x_0. Because $\nabla f(x_0)$ is not perpendicular to T we can project it onto T to get a non-zero vector v in T which is not perpendicular to ∇f. Actually the angle between ∇f and v is acute so that $\cos(\alpha) > 0$. Take a curve $r(t)$ in S with $r(0) = x_0$ and $r'(0) = v$. We have $d/dt f(r(0)) = \nabla f(r(0)) \cdot r'(0) = |\nabla f(x_0)||v| \cos(\alpha) > 0$. By linear approximation, we know that $f(r(t)) > f(r(0))$ for small enough $t > 0$. This is a contradiction to the fact that f was maximal at $x_0 = r(0)$ on S.

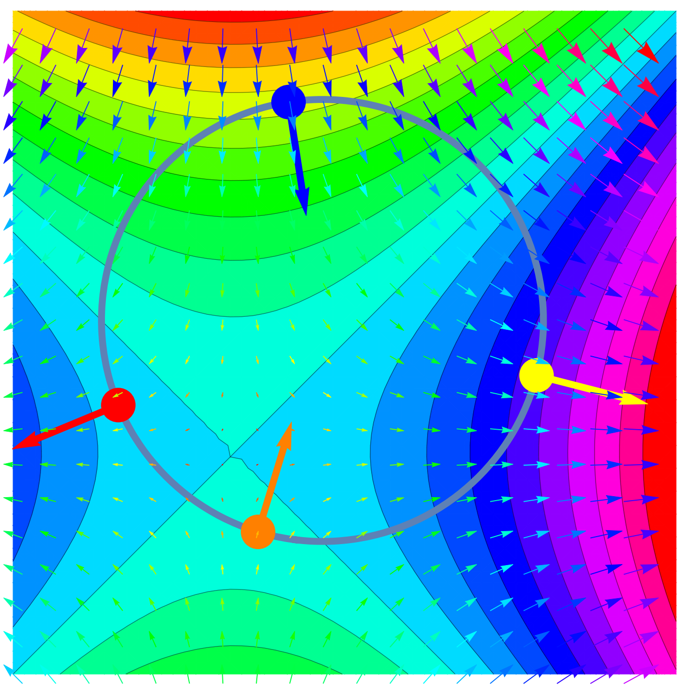


Figure 2. A Lagrange problem

20.5. This immediately implies: (distinguish $\nabla g \neq 0$ and $\nabla g = 0$)

Theorem: For a maximum of f on $S = \{g = c\}$ either the Lagrange equations $\nabla f(x_0) = \lambda \nabla g(x_0), g = c$ hold, or then $\nabla g(x_0) = 0, g = c$.

20.6. For functions $f(x, y), g(x, y)$ of two variables, this means we have to solve a system with three equations and three unknowns:

$$\begin{array}{rcl} f_x(x_0, y_0) & = & \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) & = & \lambda g_y(x_0, y_0) \\ g(x, y) & = & c \end{array}$$

20.7. To find a maximum, solve the Lagrange equations and add a list of critical points of g on the constraint. Then pick a point where f is maximal among all points. We don't bother with a second derivative test. But here is a possible statement:

$$\frac{d^2}{dt^2} D_{tv} D_{tv} f(x_0)|_{t=0} < 0$$

for all v perpendicular to $\nabla g(x_0)$, then x_0 is a local maximum.

20.8. Of course, the case of maxima and minima are analog. If f has a maximum on $g = c$, then $-f$ has a minimum at $g = c$. We can have a maximum of f under a smooth constraint $S = \{g = c\}$ without that the Lagrange equations are satisfied. An example is $f(x, y) = x$ and $g(x, y) = x^3 - y^2$ shown in Figure (3).

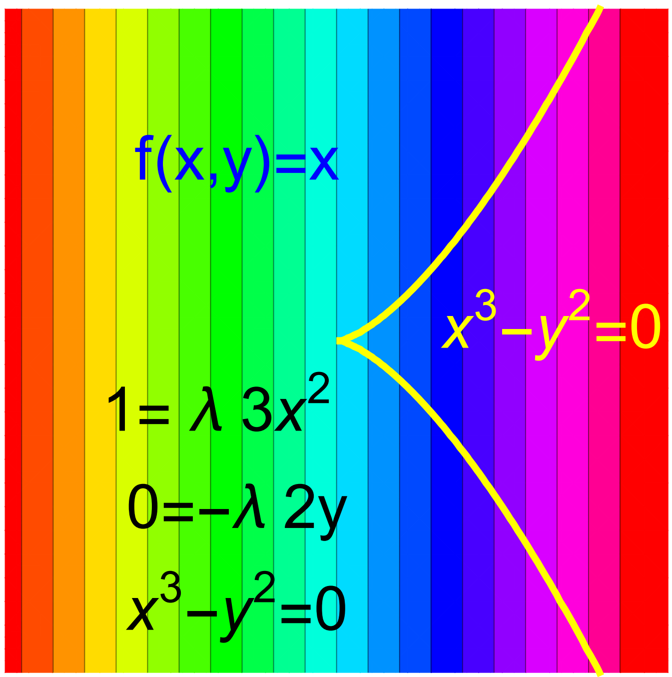


Figure 3. An example of a function, where the Lagrange equations do not give the minimum, here $(0, 0)$. It is a case, where $\nabla g = 0$.

20.9. The method of Lagrange can maximize functions f under several constraints. Lets show this in the case of a function $f(x, y, z)$ of three variables and two constraints $g(x, y, z) = c$ and $h(x, y, z) = d$. The analogue of the Fermat principle is that at a maximum of f, the gradient of f is in the plane spanned by ∇g and ∇h. This leads to the **Lagrange equations** for 5 unknowns x, y, z, λ, μ.

$$\begin{array}{rcl} f_x(x_0, y_0, z_0) & = & \lambda g_x(x_0, y_0, z_0) + \mu h_x(x_0, y_0, z_0) \\ f_y(x_0, y_0, z_0) & = & \lambda g_y(x_0, y_0, z_0) + \mu h_y(x_0, y_0, z_0) \\ f_z(x_0, y_0, z_0) & = & \lambda g_z(x_0, y_0, z_0) + \mu h_z(x_0, y_0, z_0) \\ g(x, y, z) & = & c \\ h(x, y, z) & = & d \end{array}$$

20.10. For example, if $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = x^2+y^2 = 1$, $h(x,y,z) = x+y+z = 4$, then we find points on the ellipse $g = 1, h = 4$ with minimal or maximal distance to 0.

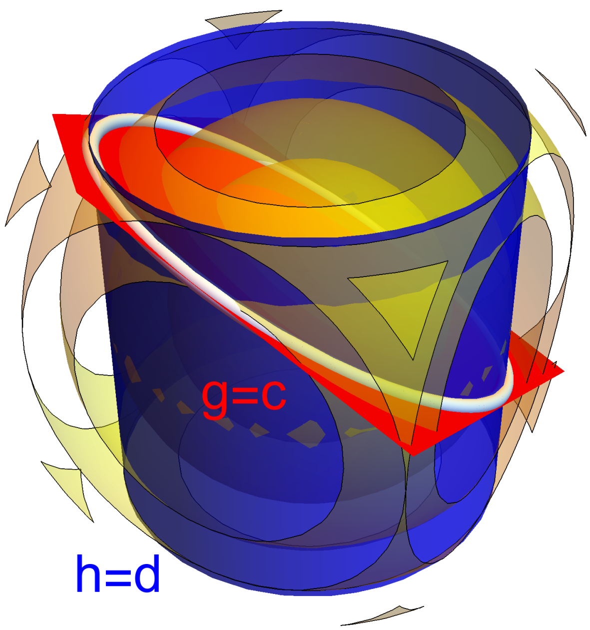


Figure 4. We see a situation where we try to maximize a function f under two constraints. In this case the intersection $g = c, h = d$ is an ellipse.

Examples

20.11. Problem: Minimize $f(x,y) = x^2+2y^2$ under the constraint $g(x,y) = x+y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

20.12. Problem: Which cylindrical soda can of height h and radius r has minimal surface A for fixed volume V? **Solution:** We have $V(r,h) = h\pi r^2 = 1$ and $A(r,h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x,y) = 2xy + 2\pi y^2$ under the constrained $g(x,y) = xy^2 = 1$. We will do that in class.

20.13. Problem: If $0 \leq p_k \leq 1$ is the probability that a dice shows k, then we have $g(p) = p_1 + p_2 + \cdots + p_6 = 1$. This vector p is called a **probability distribution**. The **Shannon entropy** of p is defined as

$$S(p) = -\sum_{i=1}^{6} p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - \cdots - p_6 \log(p_6) .$$

Find the distribution p which maximizes entropy S. **Solution:** $\nabla f = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + \cdots + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda + 1)) = 6\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_1 = p_2 = \cdots = p_6 = 1/6$. It is the fair dice that has maximal entropy. Maximal entropy means **least information content**.

20.14. Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k. Nature minimizes the **free energy**

$$F(p_1, \dots, p_n) = -\sum_i [p_i \log(p_i) - E_i p_i]$$

if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1 - \lambda)$. The constraint $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The **Gibbs solution** is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$. [1]

20.15. If f is a quadratic function on $\mathbb{R}^m$ and g is linear that is $f(x) = Bx \cdot x/2$ with $B \in M(m,m)$ and if the constraint $g(x) = Ax = c$ is linear $A \in M(1,m)$, then $\nabla f(x) = Bx$ and $\nabla g(x) = A^T$. Lets call $b = A^T \in M(m,1) \sim \mathbb{R}^m$. The Lagrange equations are then $Bx = \lambda b, Ax = c$. We see in general that for quadratic f and linear g, we end up with a **linear system of equations.**

20.16. Related to the previous remark is the following observation. It is often possible to reduce the Lagrange problem to a problem without constraint. This is a point of view often taken in economics. Let us look at it in dimension 2, where we extremize $f(x,y)$ under the constraint $g(x,y) = 0$. Define $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$. The Lagrange equations for f, g are now equivalent to $\nabla F(x,y,\lambda) = 0$ in three dimensions.

[1]This example is from Rufus Bowen, Lecture Notes in Math, 470, 1978

Homework

Problem 20.1: Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to maximize $f(x,y) = \pi x^2 y$ under the constraint $g(x,y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

Problem 20.2: Given a $n \times n$ symmetric matrix B, we look at the function $f(x) = x \cdot Bx$. and look at extrema of f under the constraint that $g(x) = x \cdot x = 1$. This leads to an equation

$$Bx = \lambda x \ .$$

A solution x is called an **eigenvector**. The Lagrange constant λ is an **eigenvalue**. Find the solutions to $Bx = \lambda x, |x| = 1$ if B is a 2×2 matrix, where $f(x,y) = ax^2 + (b+c)xy + dy^2$ and $g(x,y) = x^2 + y^2$. Then solve the problem with $a = 4, b = 1, c = 1, d = 4$.

Problem 20.3: Which pyramid of height h over a square $[-a,a] \times [-a,a]$ with surface area is $4a\sqrt{h^2+a^2} + 4a^2 = 4$ has maximal volume $V(h,a) = 4ha^2/3$? By using new variables (x,y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x,y) = yx^2$ over the constraint $g(x,y) = x\sqrt{y^2+x^2} + x^2 = 1$. Use the later variables.

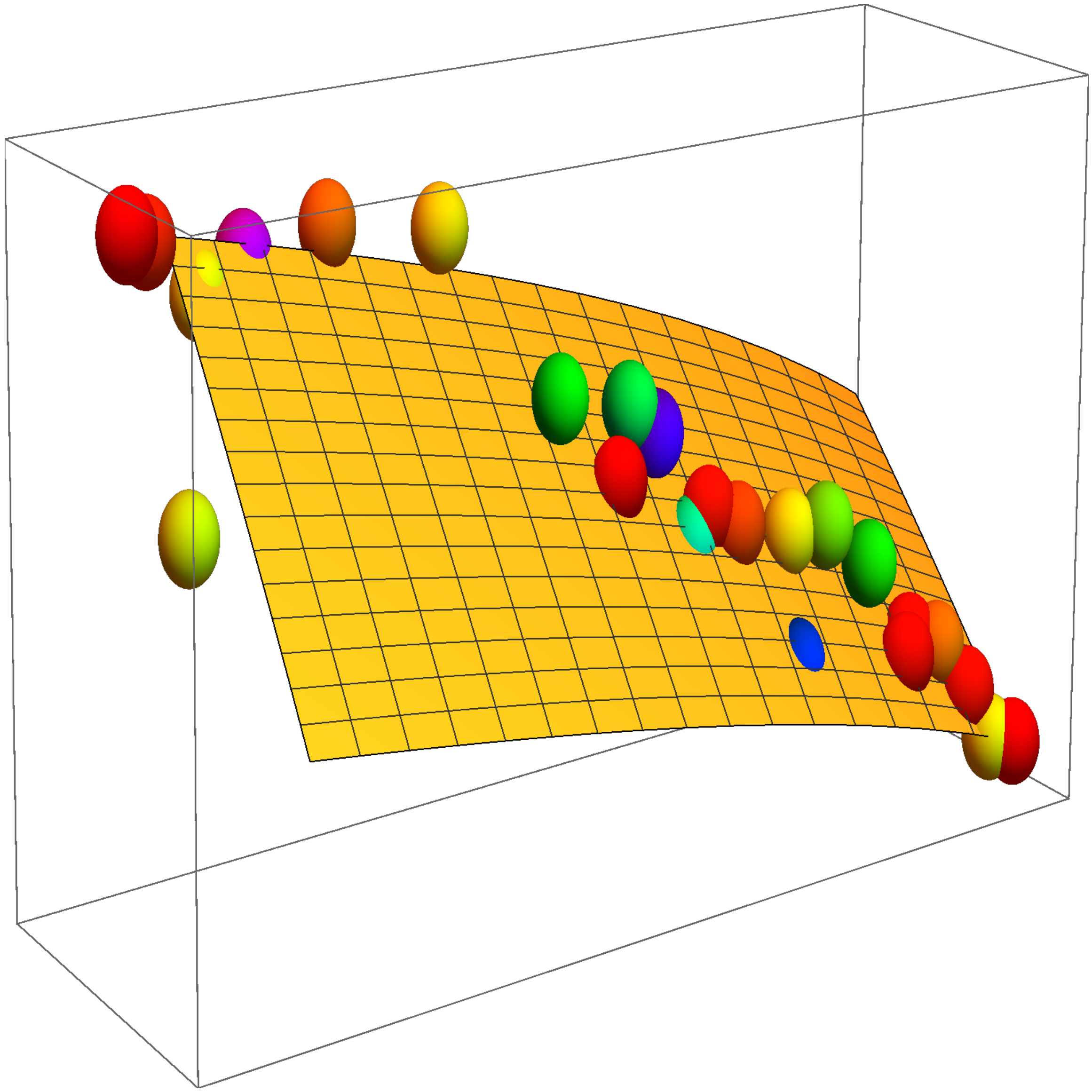
Problem 20.4: Motivated by the Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x,y,z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x,y,z) = xyz$.

Problem 20.5: A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh + \pi r^2$. Doctor Manhattan designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes $f(h,r) = 3r^2 + 2rh$ under the constraint $g(h,r) = 2r^3 + 3r^2 h = 1$. Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

Appendix: Data illustration: Cobb Douglas

20.17. The mathematician and economist **Charles W. Cobb** at Amherst college and the economist and politician **Paul H. Douglas** who was also teaching at Amherst, found in 1928 empirically a formula $F(K,L) = L^{\alpha}K^{\beta}$ which fits the **total production** F of an economic system as a function of the **capital investment** K and the **labor** L. The two authors used logarithms variables and assumed linearity to find α, β. Below are the data normalized so that the date for year 1899 has the value 100.

$Year$	K	L	P
1899	100	100	100
1900	107	105	101
1901	114	110	112
1902	122	118	122
1903	131	123	124
1904	138	116	122
1905	149	125	143
1906	163	133	152
1907	176	138	151
1908	185	121	126
1909	198	140	155
1910	208	144	159
1911	216	145	153
1912	226	152	177
1913	236	154	184
1914	244	149	169
1915	266	154	189
1916	298	182	225
1917	335	196	227
1918	366	200	223
1919	387	193	218
1920	407	193	231
1921	417	147	179
1922	431	161	240



The graph of $F(L,K) = L^{3/4}K^{1/4}$ fits pretty well that data set. You can see in the data that there is an out-layer.

20.18. Assume that the labor and capital investment are bound by the additional constraint $G(L,K) = L^{3/4} + K^{1/4} = 50$. (This function G is unrelated to the function $F(L,K)$ as we are in a Lagrange problem.) Where is the production P maximal under this constraint? Plot the two functions $F(L,K)$ and $G(L,K)$.