Unit 25: Solids

Introduction

25.1. 1-dimensional objects are curves and 2-dimensional objects are regions or surfaces. In dimension 3, we deal with **solids**. The simplest solids imaginable are the cube or the spherical ball. Solids in three dimensional space are usually drawn by plotting their boundary surfaces. A solid polyhedron for example is bound by planes. The first figure shows the solid bound by hyperboloids. It is quite a challenge to compute its volume. [1]

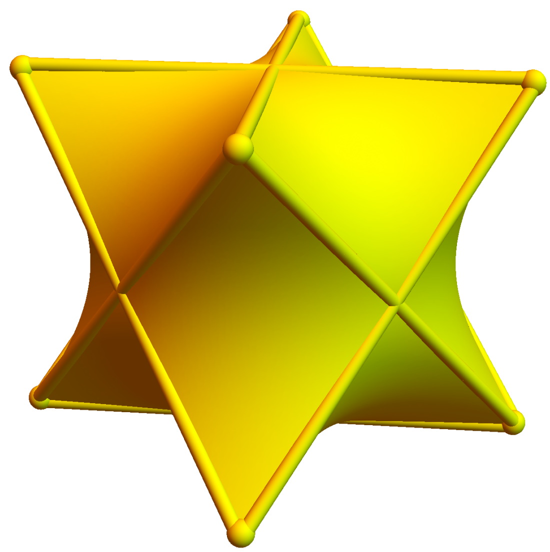

Figure 1. The "Archimedes revenge problem" asks to prove that $E : x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1$ has $\mathrm{Vol}(E) = \log(256)$.

25.2. While curves C have **length** and regions S have **area**, three dimensional solids E have **volume**. We will in the next lecture look at surface area $\int\int_S 1 \; dS$. In this lecture we look at volume $\int\int\int_E 1 \; dV$

[1] Archimedes Revenge, first appeared in Math S21a exam, Harvard Summer School, 2017

Lecture

25.3. A **basic solid** R in $\mathbb{R}^n$ is a bounded region enclosed by finitely many surfaces $g_i(x_1, \cdots, x_n) = c_i$. A **solid** is a finite union of such basic solids. We focus here mostly on $n = 3$. A 3D integral $I = \iiint_R f(x, y, z)\ dxdydz$ is defined in the same way as a limit of a Riemann sum I_n which for a given integer n is defined as

$$I_n = \frac{1}{n^3} \sum_{(i/n,j/n,k/n)\in R} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) .$$

The convergence is proven in the same way. The boundary contribution can be neglected in the limit $n \to \infty$. If $\Phi : R \to E$ is a parametrization of the solid, then

Theorem: $\iiint_R f(u, v, w)|d\Phi(u, v, w)|dudvdw = \iiint_E f(x, y, z)\ dxdydz$

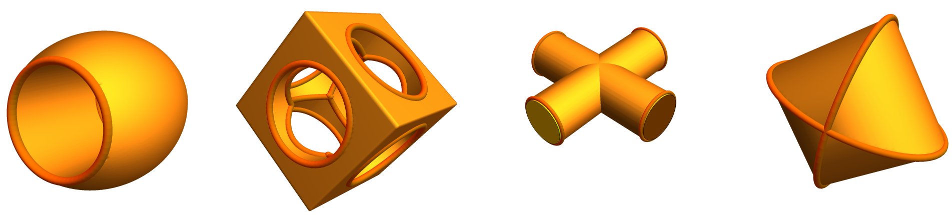

Figure 2. Solids in $\mathbb{R}^3$ are sets which are unions of solids bound by smooth surfaces. The second solid appears in homework 25.3, the last in 25.2

25.4. If $f(x, y, z)$ is constant 1, then $\iiint_E f(x, y, z)\ dxdydz$ is the **volume** of the solid E. For a cone $x^2 + y^2 \leq z^2, 0 \leq z \leq 1$, we can write $\iiint 1\ dzdxdy = \iint_R 1 - \sqrt{x^2 + y^2}\ dxdy$, where R is the unit disc. Its volume is $\pi - 2\pi/3 = \pi/3$. For the unit sphere $x^2 + y^2 + z^2 \leq 1$ for example, we can write $\iiint_E 1\ dzdxdy = \iint_R 2\sqrt{1 - x^2 - y^2}\ dxdy$, where R is the unit disc $x^2 + y^2 \leq 1$. In polar coordinates, we get $\int_0^{2\pi} \int_0^1 2\sqrt{1 - r^2} r\ drd\theta = 4\pi/3$. We can also use spherical coordinates $\Phi([\rho, \phi, \theta]) = [\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)]$, where $|d\Phi| = \rho^2 \sin(\phi)$. The volume is $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin(\phi)\ d\rho d\phi d\theta = 4\pi/3$.

25.5. There are two basic strategies to compute the integral: the first is to slice the region up along a line like the z-axis then form $\int_a^b \iint_{R(z)} f(x, y, z)dxdydz$. To get the volume of a cone for example, integrate $\int_0^1 [\iint_{R(z)} 1dxdy]dz$. The inner double integral is the area of the slice which is πz^2. The last integral gives $\pi/3$. A second reduction is to see the solid sandwiched between two graphs of a function on a region R, then form $\iint_R [\int_{g(x,y)}^{h(x,y)} f(x, y, z)\ dz]dxdy$. In the cone case, we have for R the disc of radius 1. The lower function is $g(x, y) = \sqrt{x^2 + y^2}$ the upper function is 1. We get $\iint_R [1 - \sqrt{x^2 + y^2}]\ dxdy$, a double integral which best can be computed using polar coordinates: $\int_0^{2\pi} \int_0^1 (1 - r)rdrd\theta = 2\pi(1/2 - 1/3) = \pi/3$. Burgers and fries!

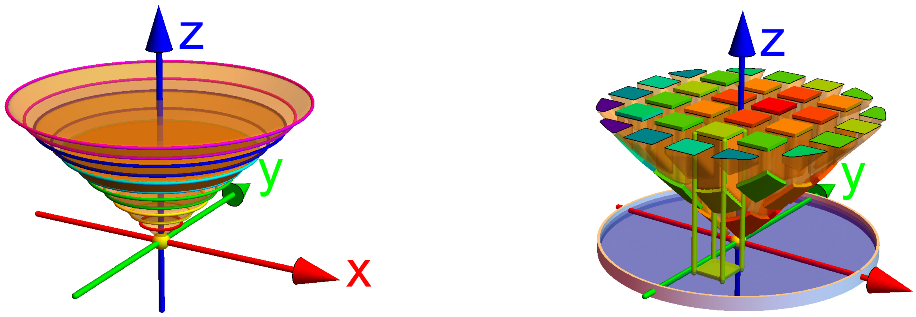

FIGURE 3. The "burger and fries methods" to compute triple integral. The first reduces to a single integral, the second to a double integral.

25.6. We have seen in the theorem the coordinate change formula if $\Phi : R \to E$ is given. For **spherical coordinates** $\Phi([\rho, \phi, \theta]) = [\rho \sin(\phi)\cos(\theta), \rho \sin(\phi)\sin(\theta), \rho \cos(\phi)]$, we have $\boxed{|d\phi| = \rho^2 \sin(\phi)}$. For **cylindrical coordinates**, the situation is the same as for polar coordinates. The map $\Phi([r, \theta, z]) = [r\cos(\theta), r\sin(\theta), z]$ produces $\boxed{|d\Phi| = r}$.

25.7. Let us find the integral $\iiint_E 1\ dxdydz$, where $E = \{x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$ is a **solid ellipsoid**. The most comfortable way is to introduce another coordinate change $\Psi([x, y, z]) \to [ax, by, cz]$ which maps the solid sphere S to to the solid ellipsoid E. Then take the spherical coordinate map $\phi : R \to S$, where $R = \{(\rho, \phi, \theta) \mid 0 \leq \rho \leq 1, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$. Now $\Psi \circ \Phi : R \to E$ is a coordinate change which maps R to the ellipsoid. By the chain rule, the distortion factor is $|d\Psi||d\Phi| = abc\rho^2 \sin(\phi)$. The integral is $abc(1/3)(2\pi)\int_0^\pi \sin(\phi)\ d\phi = \boxed{(4\pi/3)(abc)}$.

25.8. In order to compute the volume of a **solid torus**, we can introduce a special coordinate system $\Phi([r, \psi, \theta]) = [(b + ar\cos(\psi))\cos(\theta), (b + ar\cos(\psi))\sin(\theta), a\sin(\psi)]$. The solid torus E is then the image of the cuboid $\{(r, \psi, \theta) \mid 0 \leq r \leq 1, 0 \leq \psi \leq 2\pi, 0 \leq \theta \leq 2\pi\}$. The determinant is $|d\Phi| = a^2\cos^2(s)(b + ar\cos(s))$. Integration over the cuboid gives the volume $(2\pi b)(\pi a^2)$.

EXAMPLES

25.9. To find $\iiint_E f\ dV$ for $E = \{0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ and $f(x, y, z) = 24x^2y^3z$, set up the integral $\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx$. Start with the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1.

25.10. To find the **moment of inertia** $I = \iiint_E x^2 + y^2 \, dV$ of a sphere $E = \{x^2 + y^2 + z^2 \leq L^2\}$, we use **spherical coordinates**. We know that $x^2 + y^2 = \rho^2 \sin^2(\phi)$ and the distortion factor is $\rho^2 \sin(\phi)$. We have therefore

$$I = \int_0^{2\pi} \int_0^{\pi} \int_0^{L} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \; d\rho d\phi d\theta = 8\pi L^5/15 \; .$$

We will see some details in class. If we rotate the sphere around the z-axis with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37}$kgm^2. With an angular velocity of $\omega = 2\pi/\text{day} = 2\pi/(86400s)$, this rotational kinetic energy is $8 \cdot 10^{37}\text{kgm}^2/(7464960000s^2) \sim 10^{29} J \sim 2.5 \cdot 10^{24}$kcal.

25.11. Problem: Find the volume E of the intersection of $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1 - x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2 \cos^2(\theta)} r \; dr d\theta$$

has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $f(x) = (1 - \sin(x)^3)\sec^2(x)$ by parts (using $\tan'(x) = \sec^2(x)$) leading to the anti-derivative $-\cos(x) + \sec(x) + \tan(x)$ of f. The result is $16 - 8\sqrt{2}$.

25.12. Problem: A **pencil** E, a hexagonal cylinder of radius 1 above the xy-plane is cut by a sharpener below the cone $z = 10 - r$. What is its volume? Solution: we consider one sixth of the pen where the base is the polar region $0 \leq \theta \leq 2\pi/6$ and $r(\theta) \leq \sqrt{3}/(\sqrt{3}\cos(\theta) + \sin(\theta))$. The pen's back is $z = 0$ and the sharpened part is $z = 10 - r$.

$$\int_0^{\pi/3} \int_0^{\sqrt{3}/(\sqrt{3}\cos(t) + \sin(t))} \int_0^{10-r} 1 \; r \; dz dr d\theta \; .$$

The integral can be computed and is a bit messy $(29 - 3\text{arctanh}(2 - \sqrt{3}))/(3\sqrt{3})$. [2]

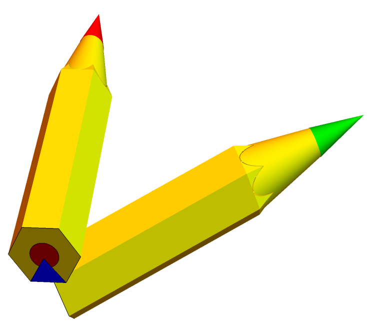

FIGURE 4. The pen problem

[2] An exam problem at ETH in a single variable calculus exam when Oliver was an undergrad.

The homework is combined in Unit 26.

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