Unit 26: Surface area

Introduction

26.1. We have looked at maps $r : R \to S$ in the context of coordinate changes and also in full generality, in the case when R is a subset of $\mathbb{R}^m$ and S is a subset of $\mathbb{R}^n$. We have learned that the Jacobian matrix dr allows to quantify the distortion $\sqrt{\det(dr^T dr)}$. If R is a subset of $\mathbb{R}^2$, then r describes a 2-dimensional surface. We usually write a point in R as (u, v) but other variables can be used. If $n = 3$, that is if we deal with a surface in three dimensional space, then the distortion factor is $|r_u \times r_v|$ and the surface area is the double integral $\iint_R |r_u \times r_v| \, dudv$. This topic is therefore a great opportunity to practice more double integrals.

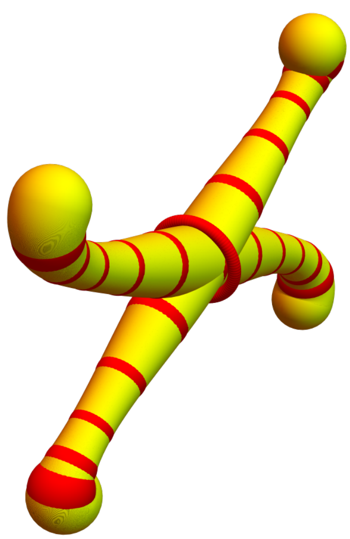

Figure 1. A circle moving in space time produces a two dimensional surface. The surface area of this surface is of interest in physics. The surface area is the **Nambu-Goto action**.

Lecture

26.2. A map $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$ has an image $r(R) = S$ which is a **parametrized surface**. What is its surface area? We have seen that the distortion factor is now $|dr| = \sqrt{\det(g)} = |r_u \times r_v|$, where $g = dr^T dr$ was the **first fundamental form** of the surface. Of course, it is more convenient to use $|r_u \times r_v|$, which is the same as $|dr|$.

Theorem: The surface area $\iint_S dS$ of S is $\iint_R |r_u \times r_v| dudv$.

26.3. More generally if $f : R \to \mathbb{R}$ is a function which describes something like a **density** then $\iint_R f(r(u,v))\ |r_u \times r_v| dudv$ is an integral which is abbreviated as $\iint_S f\ dS$ and called a **scalar surface integral**. For example, if f is a density on the surface then this $\iint_S f dS$ is the mass. Again, we have to stress that in this integral, the orientation of the surface is irrelevant. The distortion factor $|dr|$ is always non-negative. It is better to think of $\iint_S f dS$ as a **weighted surface area** generalizing area $\iint_S dS$. [1]

26.4. Here is the most general change of integration formula for maps $r : \mathbb{R}^m \to \mathbb{R}^n$, with **distortion factor** $|dr| = \sqrt{\det(dr^T dr)}$. The formula holds for $m > n$ too, det is then a pseudo determinant. If $S = r(R)$ is the image of a solid R under a C^2 map r and $f : \mathbb{R}^n \to \mathbb{R}$ is a function, then the **mother of all substitution formulas** is

Theorem: $\iint_R f(r(u))|dr(u)|\ du = \iint_S f(u)\ du$.

26.5. The proof is the same as seen in the two-dimensional change of variable situation. Just because n is used for the target space $\mathbb{R}^n$, we use the basic size $1/N$. We chop up the region into parts $R \cap Q$ with cubes Q of size $1/N$ and estimate the difference $Vol(dr(Q))$ and $Vol(r(Q))$ by CM_N/N^2 leading to an overall difference bounded by FCM_N/N^2, where F is the maximal value of f on R and M_n is the **Heine-Cantor function** modulus of continuity of f. Adding everything up gives an error $FC\text{Vol}(R)M_N + 2^n\text{Vol}(\delta R)F/N \to 0$, where δR is the boundary of R. There is one new thing: we have to see why $\sqrt{\det(A^T A)}$ is the volume of the parallelepiped spanned by the column vectors of the Jacobian matrix $A = dr$. We will talk about determinants in detail later but if A is in row reduced echelon form then $A^T A$ is the identity matrix and the determinant is 1, agreeing with the volume. Now notice that if a column of A is scaled by λ producing a new matrix B, then $\det(B^T A) = \lambda\det(A^T A)$ and $\det(B^T B) = \lambda^2\det(A^T A)$. If two columns of A are swapped leading to a new matrix B, then $\det(B^T A) = -\det(A^T A)$ and $\det(B^T B) = \det(A^T A)$. If a column of A is added to another column, then this does change $\det(B^T B)$. The only row reduction step which affects the $|dr|$ is the scaling. But that is completely in sync what happens with the volume. QED.

26.6. The last theorem covers everything we have seen and we ever need to know when integrating scalar functions over manifolds. In the special case $n = m$ it leads to:

Theorem: $\iint_R |dr(u)|\ du = \mathbf{Vol}(S)$.

[1] Unfortunately, scalar integrals are often placed close to the integration of differential forms (like volume forms). The later are of **different nature** and use an integration theory in which spaces come with orientation. So far, if we replace $r(u,v)$ with $r(v,u)$ gives the same result (like area or mass).

26.7. Here are the important small dimensional examples:

If $m = 1, n = 3$, then $\int_a^b |r'(t)|\, dt$ is the **arc length** of the curve $C = r(I)$.

If $m = 2, n = 2$, then $\iint_R |dr|\, dudv$ is the **area** of the region $S = r(R)$.

If $m = 2, n = 3$, then $\iint_R |r_u \times r_v|\, dudv$ is the **surface area** of $S = r(R)$.

If $m = 3, n = 3$, then $\iiint_R |dr|\, dudvdw$ is the **volume** of the solid $S = r(R)$.

Examples

26.8. In all the examples of surface area computations, we take a parametrization $r(u, v) : R \to S$, then use use that the distortion factor is $\sqrt{\det(dr^T dr)} = |r_u \times r_v|$.

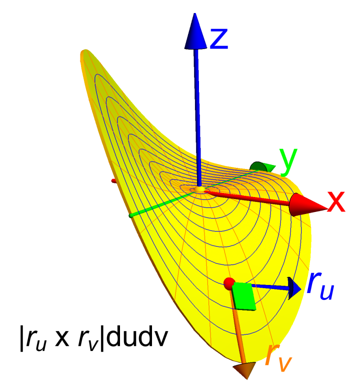

FIGURE 2. The distortion factors $|dr| = |g| = \sqrt{\det(g)} = \sqrt{\det(dr^T dr)}$ appear in general. For $m = 2, n = 3$ we get surface area $\iint_R |r_u \times r_v|\, dudv$.

26.9. Problem: find the surface area of a sphere $x^2+y^2+z^2 = L^2$. **Solution:** Parametrize the surface $r([\theta, \phi]) = [L \sin(\phi) \cos(\theta), L \sin(\phi) \sin(\theta), L \cos(\phi)]$. The distortion factor is $L^2 \sin(\phi)$. The surface area is $4\pi L^2$.

26.10. Problem: find the surface area of surface of revolution given in cylindrical coordinates as $z = g(\theta), a \leq z \leq b$. **Solution:** Parametrize the surface $r([\theta, z]) = [g(z) \cos(\theta), g(z) \sin(\theta), z]$. The distortion factor is $g(z)\sqrt{1 + g'(z)^2}$.

26.11. As an example, we can look at the surface of revolution $x^2+y^2 = 1/z^2, |z|^2 > 1$. The volume of the solid enclosed by the surface is π. The surface area is infinite.

26.12. Problem: find the surface area of the graph of a function $z = f(x, y), (x, y) \in R$. **Solution:** Parametrize the surface as $r([x, y]) = [x, y, f(x, y)]$. The distortion factor is $|r_x \times r_y| = \sqrt{1 + f_x^2 + f_y^2}$.

26.13. Problem: what is the surface area of the intersection of $x^2 + z^2 \leq 1, 6x + 3y + 9z = 12$. **Solution:** The surface is a plane but also a graph over $R = \{x^2 + z^2 \leq 1\}$ in the xz-plane. The simplest parametrization is $r([x, z]) = [x, (12 - 6x - 9z)/3, z] = [x, 4 - 2x - 3z, z]$. It gives $|r_x \times r_z| = |[-2, -1, -3]| = \sqrt{14}$. The surface area is $\iint_R \sqrt{14} dxdy = \sqrt{14}\text{Area}(R) = \sqrt{14}\pi$.

26.14. The following **hyperspherical coordinates** parametrize the 3-dimensional sphere $x^2+y^2+z^2+w^2=1$ in $\mathbb{R}^4$.

$$r([\phi,\psi,\theta]) = [\cos(\phi),\sin(\phi)\cos(\psi),\sin(\phi)\sin(\psi)\cos(\theta),\sin(\phi)\sin(\psi)\sin(\theta)] \,,$$

with $\theta \in [0,2\pi], \phi \in [0,\pi], \psi \in [0,\pi]$. The distortion factor is $\sqrt{\det(dr^T dr)} = \sqrt{\sin^4(\phi)\sin^2(\psi)}$ so that the surface area of the **hypersphere** is $2\pi\int_0^\pi\int_0^\pi \sin^2(\phi)\sin(\psi)d\phi d\psi = \boxed{2\pi^2}$.

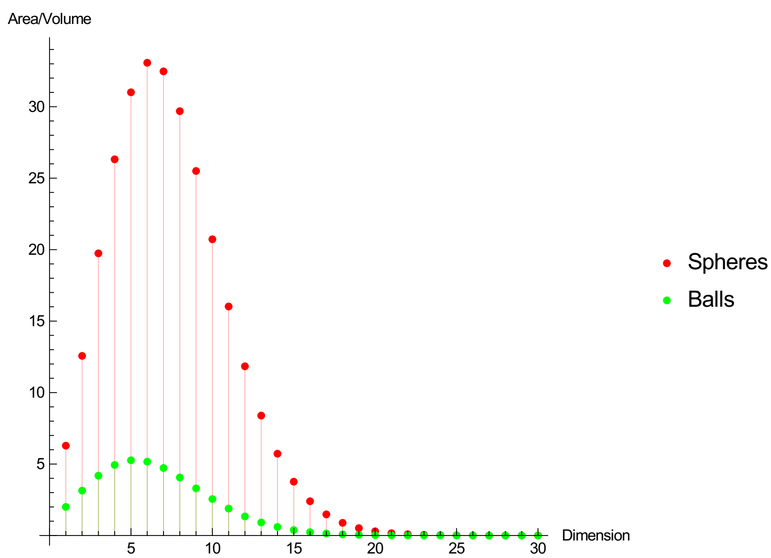

FIGURE 3. The volume and surface area of k dimensional spheres

26.15. In dimension n what is the volume $|B_n|$ of the n-dimensional **unit ball** B_n in $\mathbb{R}^n$ and the **volume** $|S_n|$ of the n-dimensional **unit sphere** S_n in $\mathbb{R}^{n+1}$? It starts with $|B_0|=1$, as B_0 is a point and $|S_0|=2$, as S_0 consists of two points. The n-ball of radius ρ has the volume $|B_n|\rho^n$ and the n-sphere of radius ρ has the volume $|S_n|\rho^n$. Because $|B_{n+1}| = \int_0^1 |S_n|\rho^n \, d\rho$, we have $|B_{n+1}| = |S_n|/(n+1)$. Because S_n can be written as a union of products $(n-2)$-spheres with S_1 leading to $|S_n| = 2\pi\int_0^{\pi/2}|S_{n-2}|\cos(\phi)\, d\phi = 2\pi|B_{n-1}|$. We know now all: just start with $|B_0|=1, |S_0|=2, |B_1|=2, |S_1|=2\pi$ and

Theorem: $|B_n| = \frac{2\pi}{n}|B_{n-2}|$, $|S_n| = \frac{2\pi}{n-1}|S_{n-2}|$.

The 5-ball has maximal volume 5.26379... among all unit balls. The 6-sphere has maximal surface area 33.0734... among all unit spheres. The volume of the 30-ball is only 0.00002.... The surface area of the 30-sphere for example is only 0.0003. Compare with a **n-unit cube** of volume 1 and a boundary surface area $2n$. High dimensional spheres and balls are tiny!

26.16. If S is a cylinder $x^2+y^2=1, 0<z<1$, triangulated with each triangle smaller than $1/n \to 0$, does the area converge to the surface area $A(S)$? No! A counter example is the **Schwarz lantern** from 1880. The cylinder is cut into m slices and n points are marked on the rim of each slice to get triangles like $A=(1,0,0), B=$

$(\cos(4\pi/n), \sin(4\pi/n, 0)), C = (\cos(2\pi/n), \sin(2\pi/n), 1/m)$ of area $\sin(2\pi/n)(1/m)\sqrt{2 + 3m^2 - 4m^2\cos(2\pi/n) + m^2\cos(4\pi/n)}/\sqrt{2}$. The nm triangles have area $\sim \sqrt{2 + 8m^2\pi^4/n^4}/\sqrt{2}$. For $m = n^3$, the triangulated area diverges.

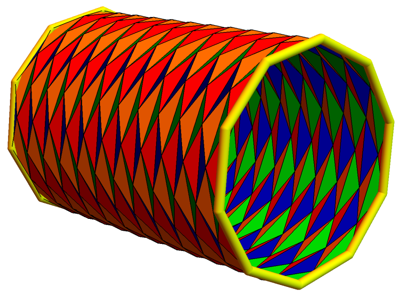

FIGURE 4. The Schwarz lantern.

26.17. The three dimensional sphere is $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. The **Hopf parametrization** is $r : R \subset \mathbb{R}^3 \to S \subset \mathbb{R}^4$ is

$$r([\phi, \theta_1, \theta_2]) = [\cos(\phi)\cos(\theta_1), \cos(\phi)\sin(\theta_1), \sin(\phi)\cos(\theta_2), \sin(\phi)\sin(\theta_2)] \ .]$$

We compute $|dr| = \sqrt{\det(dr^T dr)} = \cos(\phi)\sin(\phi) = \sin(2\phi)/2$. If we fix ϕ, we see a two dimensional torus. Their union with $\phi \in [0, \pi/2]$ is the **Hopf fibration**. We can now compute the volume of the three dimensional sphere:

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi} \sin(2\phi)/2 \; d\phi d\theta_1 d\theta_2 = 2\pi^2 \ .$$

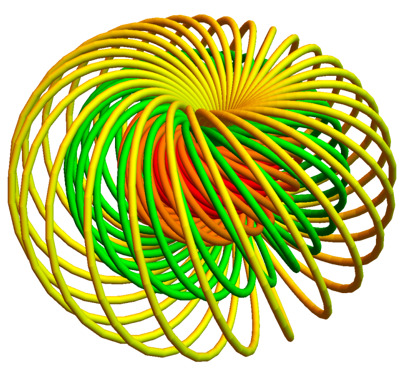

FIGURE 5. The Hopf fibration of the 3-sphere.

Homework

Problem 25-26.1: Find the moment of inertia $\iiint_E x^2+y^2\ dV$, where $E=\{x^2+y^2\leq z^2, |z|^2\leq 1$ is the double cone.

Problem 25-26.2: Evaluate the triple integral

$$\iiint_E xy\,dV\ ,$$

where E is bounded by the parabolic cylinders $y=3x^2$ and $x=3y^2$ and the planes $z=0$ and $z=x+y$.

Problem 25-26.3: We have seen the problem in the movie "Gifted" to compute the improper integral of e^{-x^2}. Here is another approach: verify

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\quad e^{-(x^2+y^2+z^2)}\ dx\,dy\,dz\ =(\sqrt{\pi})^3.$$

Use this as in the "Gifted" computation to find $\int_{-\infty}^{\infty}e^{-x^2}\ dx$. You can do that without knowing that the later is $\sqrt{\pi}$.

Problem 25-26.4: Find the surface area of the **Einstein-Rosen bridge** $r(u,v)=[3v^3, v^9\cos(u), v^9\sin(u)]^T$, where $0\leq u\leq 2\pi$ and $-1\leq v\leq 1$. Tunnels connecting different parts of space-time appear frequently in science fiction.

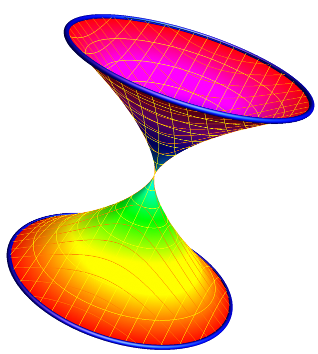

FIGURE 6. A "wormhole".

Problem 25-26.5: Find the area of the surface given by the **helicoid** $r(u,v)=[u\cos(v), u\sin(v), v]^T$ with $0\leq u\leq 1,\ \ 0\leq v\leq\pi$.