LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 29: Line integrals

Introduction

29.1. Today, we learn already how to generalize the **fundamental theorem of calculus** $\int_a^b f'(t)\ dt = f(b) - f(a)$ to higher dimensions. The interval $[a,b]$ is now replaced by a curve and the derivative $f'(t)$ becomes $\frac{d}{dt} f(r(t))$ which by the chain rule is $\nabla f(r(t)) \cdot r'(t)$. If we integrate this from a to b we get the **fundamental theorem of line integrals**.

$$\int_a^b \nabla f(r(t)) \cdot r'(t)\ dt = \int_a^b \frac{d}{dt} f(r(t)) = f(r(b)) - f(r(a)) .$$

The **gradient field** $\nabla f(x)$ can be generalized to a general vector field field $x \to F(x)$, a map which assigns to every point a vector.

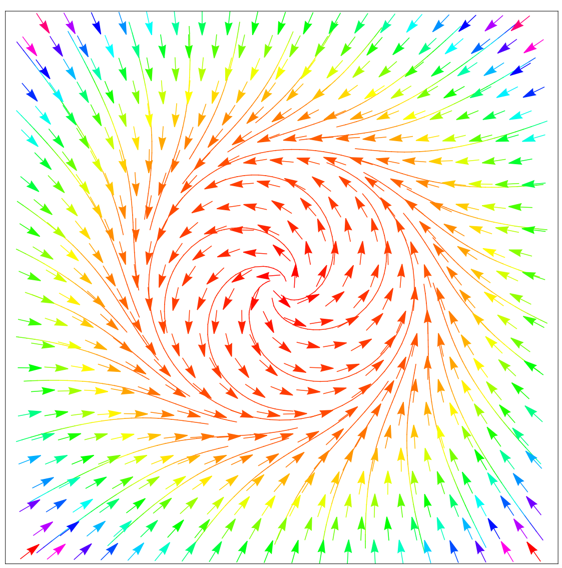

FIGURE 1. The vector field $F(x,y) = [P(x,y), Q(x,y)]^T = [x - y - x(x^2+y^2), x + y - y(x^2+y^2)]^T$ is shown with some flow lines tracing the field. In this case there exists a single flow line which is a circle. Everything gets attracted to it. It is called a limit cycle. **Hilbert 16'th problem** asked to give an upper bound for the number of possible limit cycles if P, Q are polynomials in x, y of degree n. The problem is open.

29.2. One of the questions we want to answer is under which conditions a general vector field F is a gradient field $F = \nabla f$. The reason is that if this is the case, then the integral $\int_a^b F(r(t) \cdot r'(t)\, dt$ is easy to evaluate. If F is a gradient field, the result is $f(r(b)) - f(r(a))$. In general however, vector fields are not gradient fields. In the above figure we see an example. Not all hope is lost however. We will learn in the next two weeks that in some cases, like of the path is closed, we have other ways to compute the line integral.

29.3. A good way to think about line integral is to see it as **mechanical work**. The vector field F then is thought of as a force field and the product of the force with the velocity $F \cdot r'$ is **power**, which is a scalar. Integrating power over a time gives **work**. In the case when F was a gradient field $F = \nabla f$, then f is considered a **potential energy**. The fundamental theorem of line integrals now tells that the work done over some time is just the potential energy difference. It is not really necessary to adopt this picture. The set-up is purely mathematical but in order to remember it, it can be helpful to see it associated with concepts we know. If you bike for example, then both the force applied to the pedals as well as the velocity matters.

Lecture

29.4. A **vector field** F assigns to every point $x \in \mathbb{R}^n$ a vector $F(x) = [F_1(x), \dots, F_n(x)]^T$ such that every $F_k(x)$ is a continuous function. We think of F as a **force field**. Let $t \to r(t) \in \mathbb{R}^n$ be a curve parametrized on $[a, b]$. The integral

$$\int_C F \cdot\ dr = \int_a^b F(r(t)) \cdot r'(t)\, dt$$

is called the **line integral** of F along C. We think of $F(r(t)) \cdot r'(t)$ as **power** and $\int_C F \cdot\ dr$ as the **work**. Even so F and r are column vectors, we write in this lecture $[F_1(x), \dots, F_n(x)]$ and $r' = [x_1', \dots, x_n']$ to avoid clutter. Mathematically, $F : \mathbb{R}^n \to \mathbb{R}^n$ can also be seen as a coordinate change, we think about it differently however and draw a vector $F(x)$ at every point x.

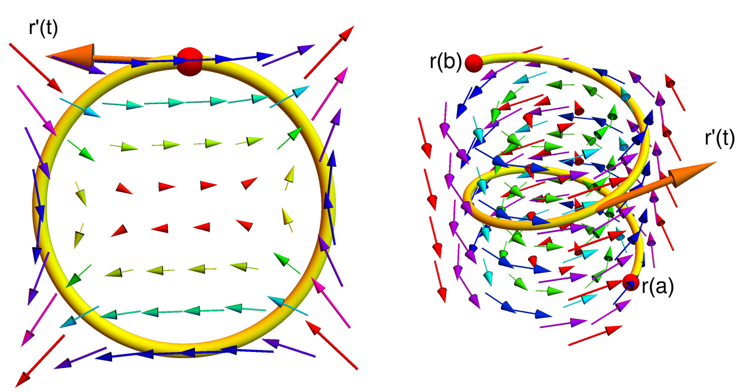

Figure 2. A line integral in the plane and a line integral in space.

29.5. If $F(x,y) = [y, x^3]$, and $r(t) = [\cos(t), \sin(t)]$ a circle with $0 \leq t \leq 2\pi$, then $F(r(t)) = [\sin(t), \cos^3(t)]$ and $r'(t) = [-\sin(t), \cos(t)]$ so that $F(r(t)) \cdot r'(t) = -\sin^2(t) + \cos^4(t)$. The work is $\int_C F \cdot \, dr = \int_0^{2\pi} -\sin^2(t) + \cos^4(t) \, dt = -\pi/4$. Figure 1 shows the situation. We go more against the field than with the field.

29.6. A vector field F is called a **gradient field** if $F(x) = \nabla f(x)$ for some differentiable function f. We think of f as the **potential**. The first major theorem in vector calculus is the **fundamental theorem of line integrals** for gradient fields in $\mathbb{R}^n$:

Theorem: $\int_a^b \nabla f(r(t)) \cdot r'(t) \, dt = f(r(b)) - f(r(a))$.

29.7. Proof: by the **chain rule**, $\nabla f(r(t)) \cdot r'(t) = \frac{d}{dt} f(r(t))$. The **fundamental theorem of calculus** now gives $\int_a^b \frac{d}{dt} f(r(t)) \, dt = f(r(b)) - f(r(a))$. QED.

29.8. As a corollary we immediately get path independence

If C_1, C_2 are two curves from A to B then $\int_{C_1} F \cdot \, dr = \int_{C_2} F \cdot \, dr$,

as well as the closed loop property:

If C is a closed curve and $F = \nabla f$, then $\int_C F \cdot \, dr = 0$.

29.9. Is every vector field F a gradient field? Lets look at the case $n = 2$, where $F = [P, Q]$. Now, if this is equal to $[f_x, f_y] = [P, Q]$, then $P_y = f_{xy} = f_{yx} = Q_x$. We see that $Q_x - P_y = 0$. More generally, we have the following **Clairaut criterion**:

Theorem: If $F = \nabla f$, then $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j = 0$.

Proof: this is a consequence of the Clairaut theorem.

29.10. The field $F = [0, x]$ for example satisfies $Q_x - P_y = 1$. It can not be a gradient field. Now, if $Q_x - P_y = 0$ everywhere in the plane, how do we find the potential f?

Integrate $f_x = P$ with respect to x and add a constant $C(y)$.

Differentiate f with respect to y and compare f_y with Q. Solve for $C(y)$.

29.11. Example: find the potential of $F(x,y) = [P, Q] = [2xy^2 + 3x^2, 2x^2y + 3y^2]$. We have $f(x,y) = \int_0^x 2xy^2 + 3x^2 \, dx + C(y) = x^3 + x^2y^2 + C(y)$. Now $f_y(x,y) = 2x^2y + C'(y) = 2x^2y + 3y^2$ so that $C'(y) = 3y^2$ or $C(y) = y^3$ and $f = x^3 + x^2y^2 + y^3$.

29.12. Here is a direct formula for the potential. Let C_{xy} be the straight line path which goes from $(0,0)$ to (x,y).

Theorem: If F is a gradient field then $f(x,y) = \int_{C_{xy}} F \cdot \, dr$.

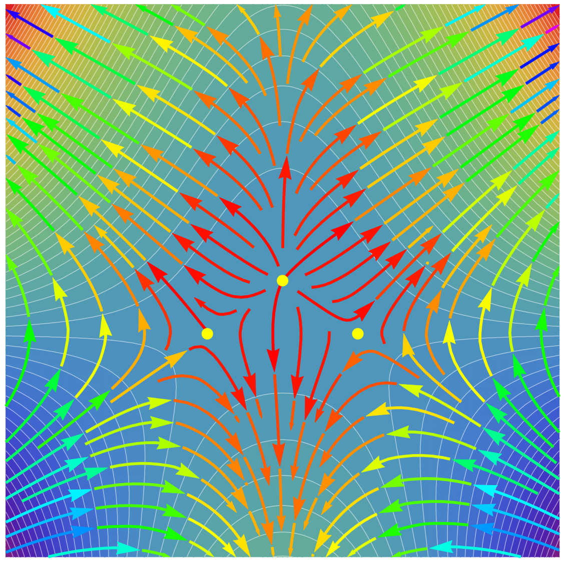

Figure 3. The vector field $F = \nabla f$ for $f(x,y) = y^2 + 4yx^2 + 4x^2$. We see the **flow lines**, curves with $r'(t) = F(r(t))$. Going with the flow **increases** f because $F(r(t)) \cdot r'(t) = |\nabla f(t)|^2$ is equal to $d/dt f(r(t))$.

29.13. Proof: By the fundamental theorem of line integral, we can replace C_{xy} by a path $[t,0]$ going from $(0,0)$ to $(x,0)$ and then with $[x,t]$ to (x,y). The line integral is $f(x,y) = \int_0^x [P,Q] \cdot [1,0] dt + \int_0^y [P,Q] \cdot [0,1]\ dt = \int_0^x P(t,0)\ dt + \int_0^y Q(x,t)\ dt$. We see that $f_y = Q(x,y)$. If we use the path going $(0,0)$ to $(0,y)$ and to (x,y) instead, the line integral is $f(x,y) = \int_0^y [P,Q] \cdot [0,1] dt + \int_0^x [P,Q] \cdot [1,0]\ dt = \int_0^y Q(0,t)\ dt + \int_0^x P(t,y)\ dt$. Now, $f_x = P(x,y)$. QED.

Examples

29.14. Find $\int_C [2xy^2 + 3x^2, 2x^2y + 3y^2] \cdot\ dr$ for a curve $r(t) = [t\cos(t), t\sin(t)]$ with $t \in [0, 2\pi]$. Answer: we found already $F = \nabla f$ with $f = x^3 + x^2y^2 + y^3$. The curve starts at $A = (1,0)$ and ends at $B = (2\pi, 0)$. The solution is $f(B) - f(A) = 8\pi^3$.

29.15. If $F = E$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r\ '(t)\ dt$ is an **electric potential**. In celestial mechanics, if F is the gravitational field, then $\int_a^b F(r(t)) \cdot r\ '(t)\ dt$ is a **gravitational potential** difference. If $f(x,y,z)$ is a temperature and $r(t)$ the path of a fly in the room, then $f(r(t))$ is the temperature, which the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(r(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference.

29.16. A device which implements a non-gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The **first law of thermodynamics** forbids the existence of such a machine. It is informative to contemplate the ideas which people have come up and to see why they don't work. We will look at examples in the seminar.

29.17. Let $F(x,y) = [P,Q] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$. Its potential $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P, f_y = (1/x)/(1+y^2/x^2) = Q$. In the seminar you ponder the riddle that the line integral along the unit circle is not zero:

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2(t)}] \cdot [-\sin(t), \cos(t)] \, dt = \int_0^{2\pi} 1 \, dt = 2\pi \ .$$

The vector field F is called the vortex.

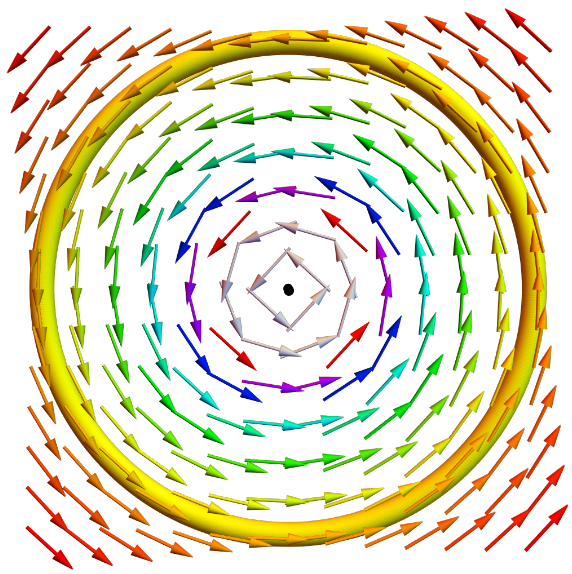

FIGURE 4. The vortex vector field has a singularity at $(0,0)$. All the curl is concentrated at $(0,0)$.

Homework

Problem 29.1: Let C be the space curve $r(t) = [\cos(t), \sin(t), \sin(t)]$ for $t \in [0, \pi/2]$ and let $F(x, y, z) = [y, x, 15]$. Calculate the line integral $\int_C F \cdot dr$.

Problem 29.2: What is the work done by moving in the force field $F(x, y) = [2x^3 + 1, 4\pi \sin(\pi y^4) y^3]$ along the quartic $y = x^4$ from $(-1, 1)$ to $(1, 1)$?

Problem 29.3: Let F be the vector field $F(x, y) = [-y, x]/2$. Compute the line integral of F along the curve $r(t) = [a\cos(t), b\sin(t)]$ with width $2a$ and height $2b$. The result should depend on a and b.

Problem 29.4: Archimedes swims around a curve $x^{22} + y^{22} = 1$ in a hot tub, in which the water has the velocity $F(x, y) = [3x^3 + 5y, 10y^4 + 5x]$. Calculate the line integral $\int_C F \cdot dr$ when moving from $(1, 0)$ to $(-1, 0)$ along the curve.

Problem 29.5: Find a closed curve $C : r(t)$ for which the vector field

$$F(x, y) = [P(x, y), Q(x, y)] = [xy, x^2]$$

satisfies $\int_C F(r(t)) \cdot r'(t)\, dt \neq 0$.

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