Unit 32: Stokes theorem

Introduction

32.1. Stokes theorem is a mountain peak in mathematics. You have not really lived before having climbed that mountain. The theorem was developed first in a physics context but it is important for other reasons. First, it is a place where many multi-variable concepts come together: it involves curves, surfaces, the dot and cross products, various derivatives like Jacobean or gradient, integrals or coordinate changes. If you master this theorem you own the bulk of this course. The theorem is also a prototype for a method in science: a theorem helps to solve problems which otherwise would be inaccessible. We will see quite many integrals which are not reachable without the theorem. Also, like mountain climbing, it produces some satisfaction top-out on something that important. The theorem is also **beautiful** $\boxed{\int_G dF = \int_{dG} F}$ and so art.

Figure 1. The **Matterhorn** in the southern part of Switzerland. Starting in the Hörnli hut (3262 meters, vectors, lines planes, curves, surfaces) one reaches the Solvay Bivouac at (4003 meters, extrema, Lagrange, integration) and arrives at the peak (4478 meters, Green, Stokes and Gauss). Image source: Wikimedia, CC BY-SA).

32.2. Proving the theorem was an exam problem given by George Stokes. James Clerk Maxwell who was a student there would later use it to formulate the **Maxwell equations** $\boxed{dF = 0, d^*F = j}$ for the **electromagnetic field** F and **charge-current** j. When **space-time** $\mathbb{R}^4$ is split into space and time, there are 4 equations. One of them is $\text{curl}(E) = -\frac{\partial}{\partial t}B/c$. It explains how an **electric potential** $\int_C E dr$ emerges from **flux changes** of a **magnetic field** B when turning a wire C, allowing us to generate **electricity** from **motion**. When reversed, it turns electricity back into **mechanical energy**. Think about Stokes theorem next time you are using an **electric motor**!

Lecture

32.3. Given a C^1 surface $S = r(G)$ in $\mathbb{R}^3$ using a parametrization $r = [x, y, z]$ and a C^1 vector field $F = [P, Q, R]$, we can form the **flux integral**

$$\iint_S F \cdot dS = \iint_G F(r(u,v)) \cdot r_u \times r_v \; dudv \; .$$

For $F = [P, Q, R]$, the **curl** is defined as $\nabla \times F = [R_y - Q_z, P_z - R_x, Q_x - P_y]$. The **Stokes theorem** tells that if $C = r(I)$ is the boundary of $S = r(G)$ and I is oriented so that G is to the left of C, then

Theorem: $\iint_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot dr$.

32.4. Proof. The key is the following **"important formula"**

$$\mathrm{curl}(F)(r(u,v)) \cdot (r_u \times r_v) = F_u \cdot r_v - F_v \cdot r_u .$$

This is straightforward and done in class. Now define the field $\tilde{F}(u,v) = [\tilde{P}, \tilde{Q}] = [F(r(u,v)) \cdot r_u(u,v), F(r(u,v)) \cdot r_v(u,v)]$ in the uv-plane. The 2-dimensional curl of $\tilde{F}$ is $\tilde{Q}_u - \tilde{P}_v = F_u \cdot r_v - F_v \cdot r_u$ as we can see by using Clairaut $r_{uv} = r_{vu}$. The Stokes theorem is now a direct consequence of **Green's theorem** proven last time. QED. [1]

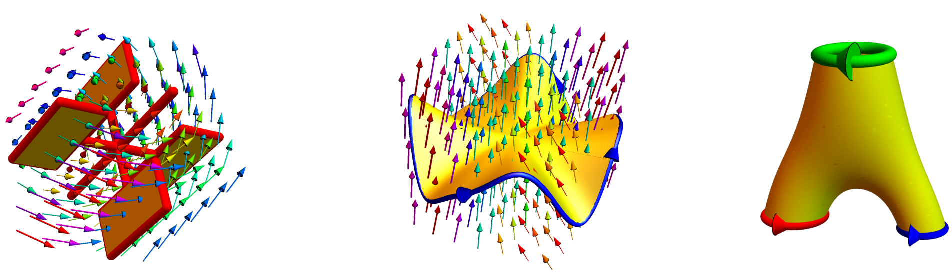

Figure 2. The paddle wheel measures curl. The boundary C has S "to the left". The pant surface illustrates a "cobordism". You definitely need to contemplate Stokes the next time you dress up your underpants!

Examples

32.5. Problem: Compute the flux of $F(x, y, z) = [0, 0, 8z^2]^T$ through the upper half unit sphere S oriented outwards. **Solution:** we parametrize the surface as $r(u, v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]^T$. Because $r_u \times r_v = -\sin(v) r$, this parametrization has the wrong orientation! We continue nevertheless and just change the sign at the end. We have $F(r(u, v)) = [0, 0, 8\cos^2(v)]^T$ so that

$$\int_0^{2\pi} \int_0^{\pi/2} -[0, 0, 8\cos^2(v)]^T \cdot [\cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)]^T \; dvdu \; .$$

[1] Mathematicians say: "we pulled back the field from $\mathbb{R}^3$ to $\mathbb{R}^2$ along the parametrization".

The flux integral is $\int_0^{2\pi}\int_0^{\pi/2}$ - $8\cos^3(v)\sin(v)\, dvdu$ which is $2\pi \cdot 8\cos^4(v)/4|_0^{\pi/2} = -4\pi$. The flux with the outward orientation is $+4\pi$. We could **not** use the Stokes theorem here because we don't deal with the flux of the curl but the flux of F itself.

32.6. Problem: What is the value of $\int_C F \cdot dr$ if $F = [\sin(\sin(x)) + z^2, e^y + x^3 + y^2, \sin(y^2) + z^2]$ and C is the unit polygon $(0,0,0) \to (1,0,0) \to (1,1,0) \to (0,1,0) \to (0,0,0)$? **Solution:** use Stokes theorem. The curl of F is $[2y\cos(y^2), 2z, 3x^2]$. The surface $S : r(u,v) = [u,v,0]$ with $0 \leq u \leq 1$ and $0 \leq v \leq 1$ has C as boundary. Stokes allows to compute $\iint_S \text{curl}(F) \cdot dS$ instead. Since $r_u \times r_v = [0,0,1]$, the flux integral is $\int_0^1\int_0^1 3u^2 dvdu = 1$. The computation of the line integral would have been more painful.

32.7. Problem: Compute the flux of the curl of $F(x,y,z) = [0,1,8z^2]^T$ through the upper half sphere S oriented outwards. **Solution:** Great, it is here, where we can use Stokes theorem $\iint_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$, where C is the boundary curve which can be parametrized by $r(t) = [\cos(t), \sin(t), 0]^T$ with $0 \leq t \leq 2\pi$. Before diving into the computation of the line integral, it is good to check, whether the vector field is a gradient field. Indeed, we see that $\text{curl}(F) = [0,0,0]$. This means that $F = \nabla f$ for some potential f implying by the **fundamental theorem of line integrals** that $\int_C F \cdot dr = 0$. But wait a minute, if the curl of F is zero, couldn't we just have seen directly that the flux of the curl through the surface is zero? Yes, we could have seen that before: for a gradient field, the flux of the curl of F through a surface is always zero, for the simple reason that the curl of such a field is zero.

32.8. Problem. What is the flux of the curl of $F(x,y,z) = [\sin(xyz), ze^{\cos(x+y)}, zx^5 + z^{22}]$ through the lower ellipsoid S given by $x^2/4 + y^2/9 + z^2/16 = 1, z < 0$? **Solution:** by Stokes theorem, it is the line integral $\int_C F \cdot dr$. Through the boundary $r(t) = [2\cos(t), 3\sin(t), 0]$. But in the xy-plane $z = 0$, the field F is zero. The result is zero.

32.9. Problem: What is the flux of the curl of F through an ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$? **Solution:** We can cut the ellipsoid into two parts to get two surfaces with boundary. The upper part $S_+ = \{(x,y,z) \in S, z > 0\}$ has the boundary $C_+ : r(t) = [2\cos(t), 3\sin(t), 0]$ which matches the orientation of the surface. Stokes theorem tells that $\iint_{S_+} \text{curl}(F) \cdot dS = \int_{C_+} F \cdot dr$. The lower part $S_- = \{(x,y,z) \in S, z < 0\}$ has the boundary $C_- : r(t) = [2\cos(t), -3\sin(t), 0]$ which matches the orientation of the lower part. Stokes theorem tells that $\iint_{S_-} \text{curl}(F) \cdot dS = \int_{C_-} F \cdot dr$. Together we have $\int_{C_-} F \cdot dr + \int_{C_+} F \cdot dr = 0$ as the line integrals have just different signs. The result is zero.

Remarks

32.10. The left hand side of the **important formula** (it "imports" the curl) [2] is defined only in three dimensions. But the right hand side also makes sense **in** $\mathbb{R}^n$. It is $\mathrm{tr}((dF)^* dr)$, where $*$ rotates the 2-frame by 90 degrees. The Stokes theorem for 2-surfaces works for $\mathbb{R}^n$ if $n \geq 2$. For $n = 2$, we have with $x(u,v) = u, y(u,v) = v$ the identity $\mathrm{tr}((dF)^* dr) = Q_x - P_y$ which is Green's theorem. Stokes has the general structure $\boxed{\int_G \delta F = \int_{\delta G} F}$, where δF is a derivative of F and δG is the boundary of G.

Theorem: Stokes holds for fields F and 2-dimensional S in $\mathbb{R}^n$ for $n \geq 2$.

32.11. Why are we interested in $\mathbb{R}^n$ and not only in $\mathbb{R}^3$? One example is that 2-dimensional surfaces appear as "paths" which a **moving string** in 11 dimension traces. More important maybe is that statisticians work by definition in high dimensional spaces. When dealing with n data points, one works in $\mathbb{R}^n$. Why would you care about theorems like Stokes in statistics? As a matter of fact, integral theorems in general allow to **simplify computations**. As we have seen in Green's theorem, when computing the sum over all the curls, there are **cancellations** happening in the inside. Integral theorems "see these cancellations" and allow to **bypass and ignore stuff which does not matter**.

32.12. The fundamental theorem of line integrals $\int_a^b \mathrm{tr}(df(r(t))dr(t))dt = f(r(b)) - f(r(a))$ holds also in $\mathbb{R}^n$. The flux integral

$$\iint_G \mathrm{tr}(F^*(r(u,v))dr(u,v)) \, dudv$$

is the analogue of a line integral in two dimensions. Written like this, we don't need the cross product. And not yet the language of **differential forms**.

32.13. Stokes deals with "fields" and "space". What happens if the field is space itself, that is if $F^* = dr$? It is of interest. For $m = 1$, and $F = dr^T$, then $\int_a^b |dr|^2 \, dt$ is the **action integral** in physics. A general **Maupertius principle** assures that it is equivalent to the **arc length** $\int_a^b |dr| \, dt$ in the sense that minimizing arc length between two points is equivalent to minimize the action integral (which is more like the energy one uses to get from the first point to the second). Now, in two dimensions we have $\iint_G \mathrm{tr}(dr^T dr) \, dudv$. We can compare this with $\iint_G \det(dr^T dr) \, dudv$ which is called the **Nambu-Goto action**, which resembles the **surface area** $\iint_G \sqrt{\det(dr^T dr)} \, dudv$ also called the **Polyakov action**. Nature likes to minimize. Free particles move on shortest paths, minimize the arc length. Maupertius tells that minimizing the length $\int_A^B |r'(t)| \, dt$ of a path equivalent to minimizing $\int_A^B r'(t) \cdot r'(t) \, dt$ which essentially is the integrated kinetic energy or gasoline use to go from A to B. For the purpose of minimizing stuff this also works for two dimensional actions. Minimizing the surface area $\iint_G |r_u \times r_v| \, dudv$ among all surfaces connecting two one dimensional curves is equivalent to minimize $\iint_G |r_u \times r_v|^2 \, dudv$. Also in higher dimensions, Nambu-Goto and Polyakov are equivalent.

[2]I learned the "important formula" from Andrew Cotton-Clay in 2009: http://www.math.harvard.edu/archive/21a_fall_09/exhibits/stokesgreen

Problem 32.1: Use Stokes to find $\int_C F \cdot dr$, where $F(x,y,z) = [12x^2y, 4x^3, 12xy+e^{(e^z)}]$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Problem 32.2: Evaluate the flux integral $\int\int_S \mathrm{curl}(F) \cdot dS$, where

$$F(x,y,z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]^T$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

Problem 32.3: Find the line integral $\int_C F\, dr$, where C is the circle of radius 3 in the xz-plane oriented counter clockwise when looking from the point $(0,1,0)$ onto the plane and where F is the vector field

$$F(x,y,z) = [4x^2z + x^5, \cos(e^y), -4xz^2 + \sin(\sin(z))]^T .$$

Use a convenient surface S which has C as a boundary.

Problem 32.4: Find the flux integral $\int\int_S \mathrm{curl}(F) \cdot dS$, where $F(x,y,z) =$

$$[y + 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz + (z-1)^{22}]^T$$

and S is the surface parametrized by

$$r(s,t) = [(1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s]^T$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.

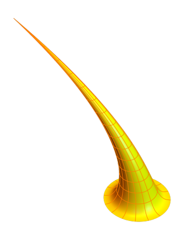

FIGURE 3. Problem 32.4 is a thorny problem! You might definitely have to discuss this with somebody else.

Problem 32.5: Assume S is the surface $x^{22} + y^8 + z^6 = 100$ and $F = [e^{e^{22z}}, 22x^2yz, x - y - \sin(zx)]$. Explain why $\int\int_S \mathrm{curl}(F) \cdot dS = 0$.

Appendix: Applications

32.14. A region E in $\mathbb{R}^n$ is called **simply connected** if it is connected and for every closed loop C in E there is a continuous deformation C_s of C **within G** such that $C_0 = C$ and $C_1(t) = P$ is a point. For example, $C(t) = [\cos(t), \sin(t), 0]$ can be deformed in $E = \mathbb{R}^3$ to a point with $C_s(t) = [(1-s)\cos(t), (1-s)\sin(t), 0]$ as $C_1(t) = P = [0,0,0]$ for all t. Each Euclidean space $\mathbb{R}^n$ is simply connected. The region $G = \{x^2+y^2 > 0\} \subset \mathbb{R}^3$ is not simply connected as the circle $C : r(t) = [\cos(t), \sin(t), 0]$ winding around the z-axis can not be pulled together to a point **within** G. The region $G = \{x^2+y^2+z^2 > 0\} \subset \mathbb{R}^3$ is simply connected, but $G = \{x^2+y^2 > 0\}$ in $\mathbb{R}^2$ is not. Remember that F was called **irrotational** if $\mathrm{curl}(F) = 0$ everywhere.

Theorem: If F is irrotational on a simply connected E then $F = \nabla f$ in E.

32.15. Proof: since E is simply connected and $\mathrm{curl}(F) = 0$, every closed loop C can be filled in by a surface $S = \bigcup_{0 \leq s \leq 1} C_s$ which has the boundary C. Stokes theorem gives $\int_S F \cdot dr = \iint_S \mathrm{curl}(F) \cdot dS = 0$. The closed loop property implies path independence. A potential f can be obtained by fixing a base point p in E, then define for any other point x a path C_{px} going from p to x. The potential function f is then defined as $f(x) = \int_{C_{px}} F \cdot dr$. QED

32.16. The field $F(x,y,z) = [-y/(x^2+y^2), x/(x^2+y^2), 0]$ is defined everywhere except on the z-axis. The domain E, where F is defined is not simply connected. There is no global function f which is a potential for F.

32.17. The notion of "simply connectedness" is important in topology. The first solved **Millenium problem**, the **Poincaré conjecture**, is now a theorem. It tells that a 3-dimensional manifold which is simply connected is topologically equivalent to the 3-sphere $\{x^2+y^2+z^2+w^2 = 1\} \subset \mathbb{R}^4$. In two dimensions, the result was known for a long time already, because the structure of 2-dimensional connected manifolds is known.

Electromagnetism

32.18. The **Maxwell-Faraday equation** in electromagnetism relates the **electric field** E and the **magnetic field** B with the partial differential equation $\mathrm{curl}(E) = -\frac{d}{dt}B$. Given a surface S, the flux integral $\iint_S B \cdot dS$ is called the **magnetic flux** of B through the surface. If we integrate the Maxwell-Faraday equation, we see that $\iint_S \mathrm{curl}(E) \cdot dS$ is equal to minus the rate of change of the magnetic flux $-\frac{d}{dt}\iint_S B \cdot dS$. Stokes theorem now assures that $\iint_S \mathrm{curl}(E) \cdot dS = \int_C E \cdot dr$ is the line integral of the electric field along the boundary. But this is **electric potential** or voltage. We see:

We can generate an electric potential by changing the magnetic flux.

32.19. Changing the magnetic flux can happen in various ways. We can generate a changing magnetic field by using **alternating current**. This is how **transformers work**. An other way to change the flux is to **rotate a wire** in a fixed magnetic field. This is the **principle of the dynamo**:

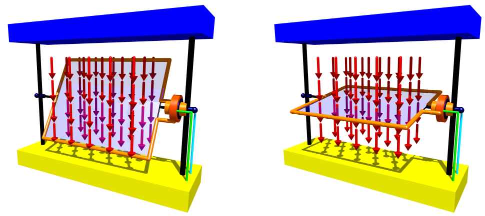

FIGURE 4. The dynamo, implemented using the ray tracer Povray. Electric current is generated by moving a wire in a fixed magnetic field.

32.20. The vector field $A(x,y,z) = \frac{[-y,x,0]}{(x^2+y^2+z^2)^{3/2}}$ is called the **vector potential** of a magnetic field $B = \text{curl}(A)$. The picture shows some flow lines of this **magnetic dipole field** B. **Problem:** Find the flux of B through the lower half sphere $x^2+y^2+z^2 = 1, z \leq 0$ oriented downwards. **Solution:** Since we have an integral of the curl of the vector field A, we use **Stokes theorem** and integrate $A(r(t))$ along the boundary curve $r(t) = [\cos(t), -\sin(t), 0]$. First of all, we have $A(r(t)) = [\sin(t), \cos(t), 0]$. The velocity is $r\,'(t) = [-\sin(t), \cos(t), 0]$. The integral is $\int_0^{2\pi} -1 \; dt = -2\pi$.

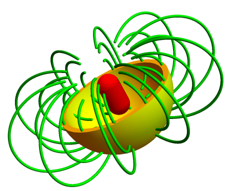

FIGURE 5. The flux of the magnetic field B through a surface can be computed with Stokes by computing a line integral of the vector potential A.

32.21. Here are all the four magical **Maxwell equations** for the **electric field** E and **magnetic field** B related to the **charge density** σ and the **electric current** j. The constant c is the speed of light. (By using suitable coordinates, one can assume $c = 1$.)

$$\boxed{\text{div}(E) = 4\pi\sigma, \text{div}(B) = 0, c \cdot \text{curl}(E) = -B_t, c \cdot \text{curl}(B) = E_t + 4\pi j}.$$

FLUID DYNAMICS

32.22. If F is the fluid velocity field and C is a closed curve, then $\int_C F \cdot dr$ is called the **circulation** of F along C. The curl of F is called the **vorticity** of F. A **vortex line** is a flow line of curl(F). Given a curve C, we can let any point in C flow along the vorticity field. This produces a **vortex tube** S. The flux of the vorticity though a surface S is the **vortex strength** of F through S. Stokes theorem implies the **Helmholtz theorem**.

Theorem: If C_s flows along F, then $\int_{C_s} F \cdot dr$ stays constant.

32.23. Proof: Let C be a closed curve and $C_s(t)$ be the curve after letting it flow using a deformation parameter s. The deformation produces a **tube surface** $S = \bigcup_{s=0}^{t} C_s$ which has the boundary C and C_t. Since the curl of F is always tangent to the surface S, the flux of the curl of F through S is zero. Stokes theorem implies that $\int_C F \cdot dr - \int_{C_s} F \cdot dr = 0$. The negative sign is because the orientation of C_s is different from the orientation of C if the surface has to be to the left.

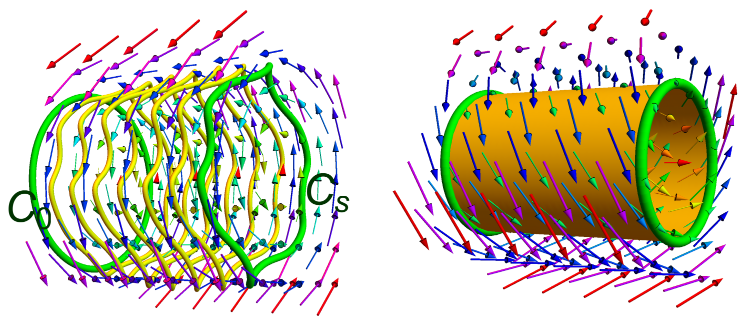

FIGURE 6. The Helmholtz theorem assures that the circulation along a flux tube is constant. This is a direct application of Stokes theorem: because the curl of F is tangent to the tube, there is no flux through the tube.

Complex analysis

32.24. An application of Green's theorem is obtained, when integrating in the complex plane $\mathbb{C}$. Given a function $f(z) = u(z) + iv(z)$ from $\mathbb{C} \to \mathbb{C}$ and a closed path C parametrized by $r(t) = x(t) + iy(t)$ in $\mathbb{C}$, define the **complex integral** $\int_a^b (u(x(t) + iy(t)) + iv(x(t) + iy(t)))(x'(t) + iy'(t))\ dt$. This is $\int_a^b u(r(t))x'(t) - v(r(t))y'(t)\ dt + i\int_a^b v(r(t))x'(t) + u(r(t))'(t)\ dt$. These are two line integrals. The real part is $F = [u, -v]$, the imaginary part is $F = [v, u]$. Assume C bounds a region G, then Green's theorem tells that the first integral is $\iint_G -v_x - u_y dxdy$ and the second integral is $\iint_G u_x - v_y\ dxdy$. It turns out now that for nice functions f like polynomials, the **Cauchy-Riemann** differential equations $\boxed{u_x = v_y, v_x = -u_x}$ hold so that these line integrals are zero. We have therefore

Theorem: If f is a polynomial and C a closed loop, $\int_C f(z)\ dz = 0$