LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 38: Review: Calculus in Hyperspace

Geometries

38.1. The **four dimensional Euclidean space** $\mathbb{R}^4 = M(4,1)$ is the space of column vectors with four real components $X = [x,y,z,w]^T$. If we think of such a vector as a **point**, we also write $X = (x,y,z,w)$. The **dot product** = **inner product** allows as usual to define **length** $|X| = \sqrt{X \cdot X}$, the **distance** $|X - Y|$ and the **angles** $\cos(\alpha) = (X \cdot Y)/(|X||Y|)$ between vectors. The **Cartesian coordinate system** has now four axes which are perpendicular to each other. Historically, as $\mathbb{R}^4$ is also the space of **quaternions**, it is custom to label the coordinate directions as $1 = [1,0,0,0], i = [0,1,0,0], j = [0,0,1,0], k = [0,0,0,1]$. A vector $[3,4,5,1]$ for example is then written also as $3 + 4i + 5j + k$. We will however keep the vector-form. We will come back in the last section of this document about why quaternions are natural.

38.2. The kernel of the 1×4 matrix $A = [a,b,c,d]$ defines the **linear hyperplane** $ax+by+cz+dw = 0$. It is a 3-dimensional linear space. An example is the **coordinate hyperplane** $x = 0$, which consists of all points $\{(0,y,z,w)\ ,\ y,z,w \in \mathbb{R}\}$. More generally, the solution space $ax + by + dz + dw = e$ is an **affine hyperplane**. The kernel of a 2×4 matrix is in general, as an intersection of two hyperplanes, a 2-dimensional plane, which we just call a **plane**. The kernel of a 3×4 matrix A is in general a **line**. Geometrically, it is the intersection of three hyperplanes.

38.3. A symmetric 4×4 matrix B, a row vector $A \in M(1,4)$ and a constant e define the **hyper quadric** $X \cdot BX + AX = e$. For a diagonal matrix $B = \mathrm{Diag}(a,b,c,d)$, this gives the quadric $ax^2+by^2+cz^2+dw^2 = e$. Examples are the **3-sphere** $x^2+y^2+z^2+w^2 = 1$, the **hyper paraboloid** $x^2 + y^2 + z^2 = w$, the 3-cylinder $x^2 + y^2 + z^2 = 1$ which is the product of a 2-sphere and a line. Or the **cylinder-plane** $x^2+y^2 = 1$ which can be seen as the product of the 1-sphere with a 2-plane. There are three types of hyperboloids like $x^2+y^2+z^2-w^2 = 1$ $x^2+y^2-y^2-z^2 = 1$ or $x^2-y^2-z^2-w^2 = 1$. One could call them **1-hyper-hyperboloids**, **2-hyper-hyperboloids** and **3-hyper-hyperboloids**, using the Morse index as a label. There is still 1-**hyperbolic-paraboloid** $x^2 + y^2 - z^2 = w$ but there are more degenerate surfaces like $x^2 - y^2 = w$. The two-dimensional torus $\mathbb{T}^2$ can be realized here as a quadratic surface. It is the intersection of $x^2 + y^2 = 1, z^2 + w^2 = 1$. This is the **flat torus**. We can not realize the two-dimensional torus in a flat way in our three dimensional space $\mathbb{R}^3$. In hyper-space, it can. There is also a **three dimensional torus** $\mathbb{T}^3$. To get a parametrization, start with the 2-torus parametrization $r(\phi,\theta) = [(3 + \cos(\phi))\cos(\theta),(3 + \cos(\phi))\sin(\theta),\sin(\phi)]$ then expand the circle to get a **hyper-torus** $r(\phi,\theta,\psi) = [(3+\cos(\phi))\cos(\theta),\ (3+\cos(\phi))\sin(\theta),\ (3+\sin(\phi))\cos(\psi),\ (3+\sin(\phi))\sin(\psi)]^T$, You see that for every fixed ψ we have a 2-torus. We can compute $4|dr| = 18 + 6\cos(\phi) + 6\sin(\phi) + \sin(2\phi)$ which is always positive and so verifies that the map from $\mathbb{T}^3$ to $\mathbb{R}^4$ is locally injective. We can also easily check that if ψ or θ is fixed we get a translated scaled version of the 2-torus. If ϕ is fixed, we get the flat 2-torus mentioned above.

38.4. In single variable calculus, one looks at graphs $\{(x,y) \mid y = f(x)\}$ of functions of one variable. In multi-variable, one adds graphs $\{(x,y,z) \mid z = f(x,y)\}$ of functions of two variables. The **graph of a function** $w = f(x,y,z)$ is now a 3-dimensional space. Paraboloids like $w = x^2+y^2+z^2$ or $w = x^2+y^2-z^2$ are graphs. An other example is the **three dimensional bell hyper-surface** $w = f(x,y,z) = \pi^{-3/2}e^{-x^2-y^2+z^2}$, where the constant has been chosen so that the **hyper-volume** $0 \leq w \leq f(x,y,z)$ is equal to 1. For obvious reasons, we usually do not draw the graph of a function of three variables as we would have to draw in 4 dimensions. Now, in hyperspace, we can do that.

38.5. Spaces can be parametrized in the same way as we parametrized curves or surfaces in three dimensions. A **curve** is defined by four real functions $x(t), y(t), z(t), w(t)$ of one variables and written as $r(t) = [x(t), y(t), z(t), w(t)]^T$. A **surface** is parametrized by $r(u,v) = [(x(u,v), y(u,v), z(u,v), w(u,v)]$. A **hypersurface** is now defined by $r(u,v,t) = [x(u,v,t), y(u,v,t), z(u,v,t), w(u,v,t)]$.

38.6. A **coordinate change** is defined by a map from $\mathbb{R}^4$ to $\mathbb{R}^4$ given by four differentiable functions: $r(u,v,s,t) = [x(u,v,s,t), y(u,v,s,t), z(u,v,s,t), w(u,v,s,t)]$. We have seen already the parametrization $r(\phi,\theta_1,\theta_0) = [\cos(\phi)\ \cos(\theta_1),\ \cos(\phi)\ \sin(\theta_1),\ \sin(\phi)\ \cos(\theta_2),\ \sin(\phi)\ \sin(\theta_2)]$ of the **unit 3-sphere= hyper-sphere** $x^2+y^2+z^2+w^2 = 1$. Because $z = x^2+y^2+z^2$ is a cylinder, there is also a natural **cylindrical coordinate system** in four dimensions. It is given by $r(\rho,\phi,\theta,w) = [\rho\sin(\phi)\cos(\theta),\ \rho\sin(\phi)\sin(\theta),\ \rho\cos(\phi), w]$. If we write down the Jacobian matrix and compute the determinant we get $\rho^2\sin(\phi)$ as in spherical coordinates.

Fields

38.7. A **scalar function** $f(x,y,z,w)$ is also called a 0-form. A **vector field** is denoted by $F = [P,Q,R,S]^T$ and a **1-form** $F = [P,Q,R,S]$ is written as $F = Pdx + Qdy + Rdz + Sdw$. A 2**-form** F has 6 components: $F = Adxdy + Bdxdz + Cdxdw + Pdydz + Qdydz + Rdzdw$. A 3**-form** again has four components $Pdydzdw + Qdxdzdw + Rdxdydw + Sdxdydz$ and a **4-form** is again completely determined by a scalar function f because $F = fdxdydzdw$.

38.8. The **exterior derivatives** are computed by using the anti-commutation rule like $dxdy = -dydx$ and $df = f_x dx + f_y dy + f_z dz + f_w dw$ and extending this to terms like $Pdydz = dPdydz = (P_x dx + P_y dy + P_z dz + P_w dw)dydz = P_x dxdydz + P_w dwdydz$. For a 1 form $F = Pdx + Qdy + Rdz + Sdw$ we have

$dF = P_x dxdx + P_y dydx + P_z dzdx + P_w dwdx + Q_x dxdy + Q_y dydy + Q_z dzdy + Q_w dwdy + R_x dxdz + R_y dydz + R_z dzdz + R_w dwdz + S_x dxdw + S_y dydw + S_z dzdw + S_w dwdw$

which simplifies to expression with 6 terms. We have $\boxed{ddF = 0}$ because every term like $P_{yz}dzdydx$ is paired with a term like $P_{zy}dydzdx$ which cancel. For a 2-form

$F = Adxdy + Bdxdz + Cdwdx + Pdydz + Qdydw + Rdzdw$, we have $dF = (A_z dz + A_w dw)dxdy + (B_y dy + B_w dw)dxdz + (C_y dy + C_z dz)dwdx + (P_x dx + P_w dw)dydz + (Q_x dx + Q_z dz)dydw + (R_x dx + R_y dy)dzdw$ which simplifies to $(Q_z + P_w + R_y)dydzdw + (B_w + C_z + R_x)dxdzdw + (A_w + Q_x + C_y)dxdydw + (A_z + B_y + P_x)dxdydz$. For a 3-form $F = Pdydzdw + Qdzdwdx + Rdwdxdy + Sdxdydz$ we have $dF = (P_x - Q_y + R_z - S_w)dxdydzdw$.

38.9. The **gradient** of a function $f(x,y,z,w)$ is defined as $\nabla f(x,y,z,w) = df^T = [f_x, f_y, f_z, f_w]^T$ The **curl** of a vector field $F(x,y,z,w) = [F_1, F_2, F_3, F_4]^T$ is the hyper-field $dF = [F_{12}, F_{13}, F_{14}, F_{23}, F_{24}, F_{34}]^T$, where we have just chosen a lexigographic order and where $F_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$. The **hypercurl** of a hyper vector field $F(x,y,z,w) = \langle F_{12}, F_{13}, F_{14}, F_{21}, F_{23}, F_{34}]$ is a 3-form but can again be associated with a vector field $dF = [F_{234}, F_{134}, F_{124}, F_{123}]^T$. The **divergence** of a vector field $F = [P, Q, R, S]$ is a 4-form $(P_x + Q_y + R_z + S_w)dxdydzdw$ but can again be associated with a scalar field.

38.10. Here are some properties which we have seen already. The gradient $\nabla f = df^T$ is perpendicular to the level surface $f(x,y,z,w) = c$. The curl of the gradient is zero. The hypercurl of the curl is zero. The divergence of the hypercurl is zero. The divergence of the gradient is the Laplacian (using the identifications, the divergence map can be identified with the adjoint $-d^*$). The **chain rule** is $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.

38.11. The **line integral** of a vector field F along a curve C is $\int_C F(r(t)) \cdot r'(t)\, dt$. The **flux integral** of a vector field F along a 2-dimensional surface is a **flux integral**. The **hyper flux integral** of a hyper-field F along a surface . The **hyper volume integral** of a function f on a solid G is $\iiiint_G f(x,y,z,w)\, dxdydzdw$.

Theorems

38.12. The **fundamental theorem of line integrals** is

> **Theorem:** $\int \nabla f(r(t)) \cdot r'(t)\, dt = f(r(b)) - f(r(a))$.

38.13. The **Stokes theorem** tells that for a surface S and 1 form F

> **Theorem:** $\iint_S \mathrm{curl}(F) \cdot dS = \int_C F \cdot dr$

38.14. The **Hyper Stokes theorem** assures that for a hypersurface S and a 2-form F, the flux of the hypercurl of F through G (a 3D-integral) is the flux of F through the boundary surface S (a 2D-integral)

> **Theorem:** $\iiint_G \mathrm{hypercurl}(F) \cdot dG = \iint_S F \cdot dS$

38.15. The **divergence theorem** assures that for a 3-form (identified as a vector field F) and a solid G with boundary hyper-surface S, we have

> **Theorem:** $\iiiint_G \mathrm{div}(F)\, dV = \iiint_S F \cdot dS$.

Quaternions

38.16. Hyperspace $\mathbb{R}^4$ is special: it is the only Euclidean space for which the unit sphere is a **non-Abelian Lie group**. A **Lie group** G is a manifold $r(\mathbb{R}^m) \subset \mathbb{R}^n$ [1] on which one has a **group operation** $x * y$ which has the property that for every y, the maps $x \to x * y$ and $x \to y * x$ are smooth maps on G. To have a **group** $(G, *)$ we must have the property that $(x * y) * z = x * (y * z)$ and that there is a **1-element** $1*x = x*1 = x$ such that every element x has an inverse x^{-1} satisfying $x*x^{-1} = 1$. The circle $\{x^2 + y^2 = 1\} = \{z \in \mathbb{C} || z| = 1\}$ is an example of a group. This multiplication is **Abelian** if $x * y = y * x$ for all $x, y \in G$. The complex plane $\mathbb{C} = \mathbb{R}^2$ is characterized as the only Euclidean space $\mathbb{R}^n$ in which the unit sphere $\mathbb{T}^1 = \{|x| = 1\}$ is an Abelian Lie group. Why Lie groups? They are the dough, **elementary particles** are baked from! Electromagnetism is built from $\mathbb{T}^1$ for example.

38.17. One can write a vector in $\mathbb{R}^4$ also as $v = a+ib+jc+kd$ where i, j, k are symbols. Hamilton noticed that when defining $i^2 = j^2 = k^2 = ijk = -1$, the 4-dimensional space becomes an algebra. An algebra is a linear space which also features a multiplication. Now one has already $M(2, 2)$, the space of 2×2 matrices, which is a 4-dimensional algebra, but the algebra which Hamilton found is a **division algebra**: every non-zero element can be inverted. This is not the case for $M(2, 2)$. The matrix in which all elements are 1 for example is non zero but it is also not invertible.

38.18. The algebra which Hamilton defined through the relations $i^2 = j^2 = k^2 = ijk = -1$ is called the **quaternion algebra** $\mathbb{H}$. If $\overline{v} = a - ib - jc - kd$, then $|v|^2 = v \cdot v = v\overline{v}$, where the right hand side is a quaternion multiplication. One can readily check that $|vw| = |v||w|$. The reason is that quaternions v can be realized as complex 2×2-matrices: if $A(v) = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$, then $|v| = \det(A(v))$ and $A(v)A(w) = A(vw)$. Your favorite AI helps to check this last identity quickly.

```
Import["Quaternions`"];
A[{x_,y_,z_,w_}]:={{x+I*y,z+I*w},{-z+I*w,x-I*y}};
Q=Quaternion[a,b,c,d]**Quaternion[p,q,r,s];
Simplify[A[{a,b,c,d}].A[{p,q,r,s}]==A[Table[Q[[k]],{k,4}]]]
```

38.19. An algebra with the property $|v * w| = |v||w|$ is a **normed division algebra**. By theorems of Hurwitz and Frobenius, there are only four: the reals $\mathbb{R}$, the complex $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. For an associative division algebra, the unit sphere is a Lie group. Because the unit sphere of $\mathbb{R}$ has only two points, the 1-circle $\{|z| = 1\} \subset \mathbb{C}$ and the unit 3-sphere $\{|z| = 1\} \subset \mathbb{H}$ are the only spheres that are Lie groups. There is a unique non-commutative one, the 3-sphere and a unique commutative connected one, the 1-sphere.

Theorem: $\mathbb{H}$ is the only non-Abelian associative normed division algebra.

Oliver Knill, knill@math.harvard.edu, Math 22b, Harvard College, Spring 2022

[1] Manifolds can be described abstractly, but a theorem of John Nash assures that every manifold can be embedded in some $\mathbb{R}^n$. So, looking at images of maps r is no loss of generality!