LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 5: Surfaces

Introduction

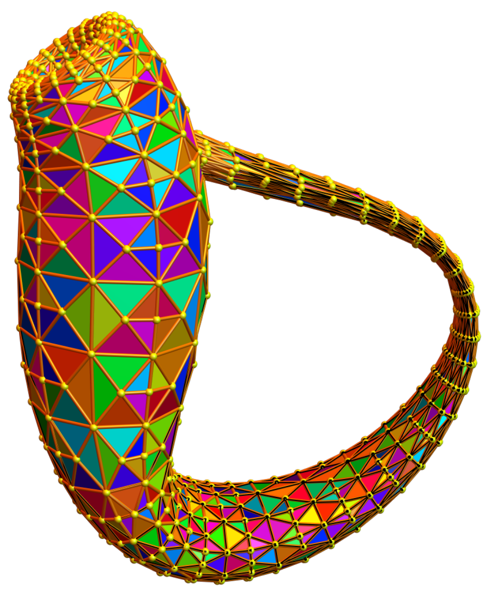

Figure 1. The Klein bottle is a surface in three dimensional space. It can not be realized as a level surface of a function however as there are points of self intersections. But we can perfectly parametrize the surface.

5.1. Surfaces are **co-dimension one** objects in a space. They are important because they can divide up space. We can confine water in a bottle. This is not possible for co-dimension two. You can not confine water in a curve. Similarly, if you lived in 4-dimensional space, you could not store water in a two-dimensional surface. But things can get tricky already in three dimensions. There are two-dimensional closed surfaces which do not confine any space. Try to drink from a Klein bottle!

5.2. A surface can mathematically be described in two fundamentally different ways. It is either given as a **level surface** of a function on that space. Or then it can be the image of a map called parametrization. You know this from the earth, which is a sphere. We can either say that a sphere is the set of points which have a fixed distance to its center point. Or then we can parametrize the sphere, for example using **longitude** and **latitude**. A plane through the 0 can be given either as the kernel $\{x \in \mathbb{R}^3 | Ax = 0\}$ of a 1×3 matrix A or then as the image $\{Ax | x \in \mathbb{R}^2\}$ of a 3×2 matrix. The first writes $ax + by + cz = 0$. The second writes the plane as $vs + wt$, where v, w are the column vectors of A and $x = [s, t]^T$ gives the parameters.

Lecture

5.3. If A is a matrix, the solution space of a system of equations $A\mathbf{x} = b$ is called a **linear manifold**. It is the set of solutions of $A\mathbf{x} = 0$ translated so that it passes through one of the points. The equation $3x + 2y = 6$ for example describes a line in $\mathbb{R}^2$ passing through $(2, 0)$ and $(0, 3)$. The solutions to $Ax = 0$ form a **linear space**, meaning that we can add or scale solutions and still have again solutions. We can rephrase the just said in that a linear space is a linear manifold which contains 0. For example, for $x+2y+3z = 6$ we get a plane which is parallel to the plane $x+2y+3z = 0$. The former is a linear manifold (also called affine space), the later is a linear space. It is the solution space to $A\mathbf{x} = 0$ with $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{x} = [x, y, z]^T$. Both planes are perpendicular to $n = [1, 2, 3]^T$. To find an equation for the plane through 3 points P, Q, R, define $n = PQ \times PR = [a, b, c]^T$ then write down $ax + by + cz = d$, where d is obtained by plugging in a point. The cross product comes handy.

5.4. The following important example deals with $A = [a_1, \dots, a_m]$ in $M(1, m)$.

Theorem: The vector $n = A^T$ is perpendicular to the plane $Ax = d$.

Proof. Given two points y, z in the plane. Then we have $Ay = d$ and $Az = d$. Then $x = y - z$ is a vector inside the plane. Now $A^T \cdot x = Ax = A(y - z) = Ay - Az = d - d = 0$. This means that x is perpendicular to the vector A^T. □

In three dimensions, this means that the plane $ax + by + cz = d$ has a normal vector $A^T = n = [a, b, c]^T$. Keep this in mind, especially because $\mathbb{R}^3$ is our home.

5.5. This **duality result** will later will identified as a **fundamental theorem of linear algebra**. It will be important in data fitting for example. The **kernel** of a matrix A is the linear space of all solution $Ax = 0$. The kernel consists of all roots of A. The **image** of a matrix A is the linear space of all vectors $\{Ax\}$. We abbreviate $\ker(A)$ for the kernel and $\mathrm{im}(A)$ of the image. We will come back to this later.

Theorem: The image of A^T is perpendicular to the kernel of A.

Proof. If x is in the kernel of A, then $Ax = 0$. This means that x is perpendicular to each row vector of A. But this means that x is perpendicular to the column vector of A^T. So, x is perpendicular to the image of A^T. This line of argument can be reversed to see that if x is perpendicular to the image of A^T, then it is in the kernel of A. □

5.6. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the solution set $\{f(x_1, \cdots, x_n) = d\}$ is a **hyper surface**. We often say "surface" even so "surface" is reserved to $n = 3$. The simplest non-linear surfaces are **quadratic manifolds**

$$x \cdot Bx + Ax = d$$

defined by a symmetric matrix B and a row vector A and a scalar d. We assume that B is not the zero matrix or else, we are in the case of a linear manifold. We also can assume B to be symmetric $B = B^T$. For notation, we write $\mathrm{Diag}(a,b,c) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $1 = \mathrm{Diag}(1,1,1)$.

5.7. Ellipsoids For $B = 1$ and $A = 0$ and $d = 1$ we get the **sphere** $|x|^2 = 1$. In $\mathbb{R}^2$, a sphere is a **circle** $x^2 + y^2 = 1$. In three dimensions we have the familiar sphere $x^2 + y^2 + z^2 = 1$. An more general ellipsoid with $B = \mathrm{Diag}(1/a^2, 1/b^2, 1/c^2)$ is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By intersecting with $x = 0$ or $y = 0$ or $z = 0$, we see **traces**, which are all ellipses.

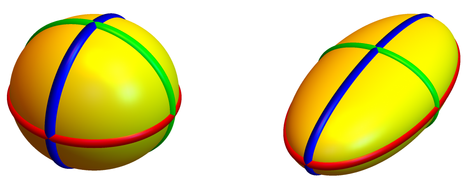

FIGURE 2. The sphere $x^2 + y^2 + z^2 = 1$ and an example of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

5.8. Hyperboloids. For $B = \mathrm{Diag}(1,1,-1)$ and $d = 1$, we get a **one-sheeted hyperboloid** $x^2 + y^2 - z^2 = 1$. For $B = \mathrm{Diag}(1,1,-1)$ and $d = -1$, we get a **two-sheeted hyperboloid** $x^2 + y^2 - z^2 = -1$. A more general hyperboloid is of the form $x^2/a^2 + y^2/b^2 - z^2/c^2 = d$ with $d \neq 0$. The intersection with $z = 0$ gives in the one-sheeted case a circle, in the two-sheeted case nothing. The $x = 0$ trace or the $y = 0$ trace are both hyperbola.

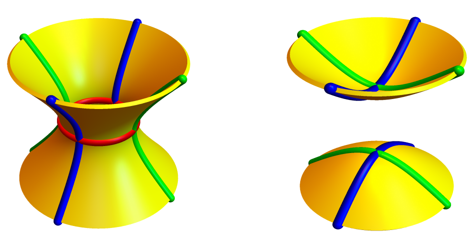

FIGURE 3. The one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ and the two-sheeted hyperboloid $x^2 + y^2 - z^2 = -1$.

5.9. Paraboloids. For $B = \mathrm{Diag}(1,1,0)$ and $A = [0,0,-1]$ and $d = 0$ we get the **paraboloid** $x^2 + y^2 = z$, for $B = \mathbf{Diag}(1,-1,0)$ and $A = [0,0,-1]$ and $d = 0$ we get the **hyperbolic paraboloid** $x^2 - y^2 = z$. We can recognize paraboloids by intersecting with $x = 0$ or $y = 0$ to see parabola. Intersecting the elliptical paraboloid $x^2 + y^2 = z$ with $z = 1$ gives an ellipse. Intersecting the hyperbolic paraboloid $x^2 - y^2 = z$ with $z = 1$ gives a hyperbola.

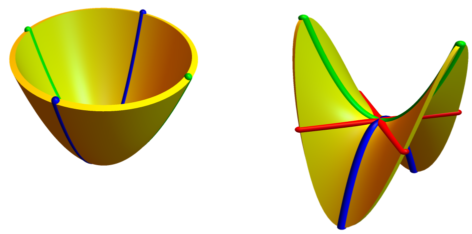

FIGURE 4. An elliptic paraboloid $z = x^2 + y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

5.10. Special surfaces. If $B = \text{Diag}(1, 1, -1)$ and $d = 0$, we get a **cone** $x^2+y^2-z^2 = 0$. For $B = \text{Diag}(1, 1, 0)$ and $d = 1$ we get the **cylinder** $x^2 + y^2 = 1$.

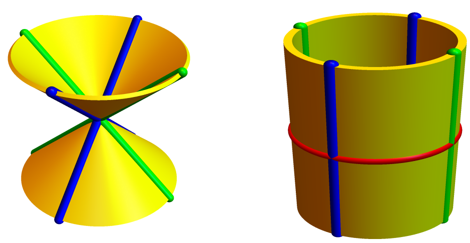

FIGURE 5. The cone $x^2 + y^2 = z^2$ and the cylinder $x^2 + y^2 = 1$.

5.11. Side remark: The 1-sphere $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ and the 3-sphere $S^3 = \{x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$ carry a multiplication: S^1 is in the **complex numbers** $\mathbb{C} = \{x+iy\}$ and S^3 is in the **quaternions** $\mathbb{H} = \{x+iy+jz+kw\}$. The 1-sphere is the gauge group for **electromagnetism**, the 3-sphere (also called $SU(2)$) is responsible for the **weak force**. No other Euclidean sphere carries a multiplication for which $x \to x * y$ is smooth. Michael Atiyah once pointed out that this algebraic particularity might not be a coincidence and responsible for the structure of the **standard model of elementary particles** (one of the most accurate theories ever built by humanity). The strong force appears as one can let a set of 3×3 matrices $SU(3)$ act on $\mathbb{H}$. Atiyah suggested that gravity could be related to the **octonions** $\mathbb{O}$. There $S^7 = \{|x| = 1\} \subset \mathbb{R}^8$ carries still a multiplication, but it is no more associative. The list of normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. [1]

5.12. Given a polynomial p of n variables, one can look at the surface $\{p(x) = 0\}$. It is called a **variety**.

[1] See the talk of 2010 of Atiyah ($https://www.youtube.com/watch?v=zCCxOE44M\_M$).

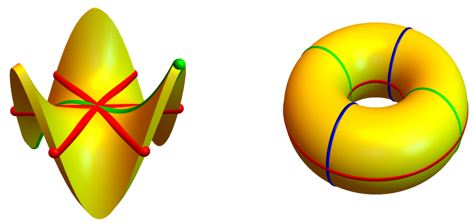

Figure 6. More examples of **varieties**, solution sets to polynomial equations. To the left we see **cubic surface** $x^3 - 3xy^2 - z = 0$ called the **monkey saddle**. To the right we see **torus** $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$ which is an example of a **quartic manifold**.

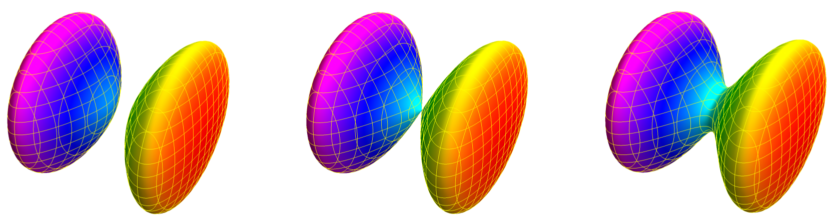

Figure 7. The variety $x^4 - x^2 + y^2 + z^2 = d$ for $d = -0.02, d = 0$ and $d = 0.02$.

Examples

5.13. Q: Find the plane Σ containing the line $x = y = z$ and the point $P = (3, 4, 5)$. A: Σ contains $Q = (0, 0, 0)$ and $R = (1, 1, 1)$ and so the vectors $v = [1, 1, 1]^T$ and $w = [3, 4, 5]^T$. The cross product between v and w is $[1, -2, 1]^T$. It is perpendicular to Σ. So, the equation is $x - 2y + z = d$, where d can be obtained by plugging in a point $(3, 4, 5)$. This gives $d = 0$ so that $x - 2y + z = 0$.

5.14. Can we identify the surface $x^2 + 2x + y^2 - 4y - z^2 + 6z = 0$? **Completion of the square** gives $x^2 + 2x + \boxed{1} + y^2 - 4y + \boxed{4} - z^2 + 6z - \boxed{9} = \boxed{1} + \boxed{4} - \boxed{9} = -4$. Now $(x + 1)^2 + (y - 2)^2 - (z - 3)^2 = -4$. This is a two-sheeted hyperboloid centered at $(-1, 2, 3)$.

5.15. Intersecting the **cone** $x^2 + y^2 = z^2$ with the plane $y = 1$ gives a hyperbola $z^2 - x^2 = 1$. Intersection with $z = 1$ gives a circle $x^2 + y^2 = 1$. Intersecting with $z = x + 1$ gives $y^2 = 2x + 1$, a parabola. Because bisecting a cone can give hyperbola, an ellipse or a parabola as cuts, one calls the later **conic sections**.

5.16. The case of **singular quadratic manifolds** is even richer: $x^2 - y^2 = 1$ is a **cylindrical hyperboloid**, $x^2 - y^2 = 0$ is a union of two planes $x - y = 0$ and $x + y = 0$. The surface $x^2 = 1$ is a union of two parallel planes, the surface $x^2 = 0$ is a plane.

Homework

Problem 5.1: a) What kind of curve is $2x^2 + 4x + 2y^2 + 2 = 0$?
b) What surface is $x^2 + y^2 - 4y + z^2 + 8z = 100$?
c) Let (x, y, z) be the set of points for which $|[x, y, z]^T \times [1, 1, 1]^T| = 1$. Describe this set.

Problem 5.2: a) What kind of curves can you get when you intersect hyperbolic paraboloid $x^2 - y^2 = z$ with a plane? b) Explore what you get if you intersect the hyperboloid $S : x^2 + y^2 - z^2 = 1$ with the S rotated by 90 degrees around the x-axes.

Problem 5.3: Find explicit planes which when intersected with the hyperboloid $x^2 + 2y^2 - z^2 = 1$ produces an ellipse, or a hyperbola or a parabola.

Problem 5.4: Find the equation of a plane which is tangent to the three unit spheres centered at $(3, 4, 5), (1, 1, 1), (2, 3, 4)$.

Problem 5.5: Build a concrete function $f(x, y, z)$ of three variables such that some level surface $f(x, y, z) = c$ is a **pretzel**, a surface with three holes. Hint: the surface $g * h = 0$ is the union of the surfaces $g = 0$ and $h = 0$. Now, $g * h = c$ can produce surfaces in which things are glued nicely. If you should look up a surface on the web or literature, you have to give the reference. You can use the computer to experiment, or then describe your strategy in words.


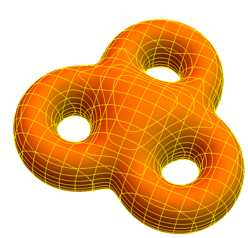

Figure 8. In the pretzel baked to the right we have used a polynomial $f(x, y, z)$ of degree 12. A problem in algebraic geometry would be to find the "smallest degree polynomial" which works and then find the most elegant polynomial.