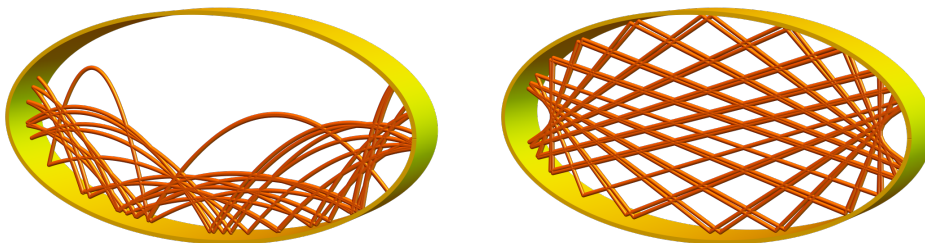# Unit 7: Curves

INTRODUCTION

FIGURE 1. The trajectory of a ball bouncing in an elliptical barrel is
a curve. It is a one dimensional object because it can be described by
one parameter. In this case we have a curve which is continuous but not
smooth. Still, we can use calculus to describe properties of the curve like
to see that it consists of pieces of parabola. By the way, on the left, with
gravity, this is a system we do not understand. The bouncing ball moves
chaotically. We do not have the tools for example to tell where the ball
is after 10^{100} bounces. To the right we see the situation without gravity.
Now, we could determine where the ball is after 10^{100} bounces.

7.1. Many geometric objects can be assigned a **dimension**. This number tells how
many parameters we need to describe the object. A point has dimension 0, a line has
dimension 1, a plane has dimension 2. This is formalized in **linear algebra**. Given
a matrix A, the number of leading 1 in rref(A) is the dimension of the image of A.
The number of free variables (columns without leading 1 in rref(A) is the dimension
of the kernel of A. For example, for $A = [1, 2, 3]$ which is already in row reduced form,
we have one leading 1 and two free variables y and z. The equation 1 x + 2 y + 3
z = 0 describes a 2 dimensional object, a plane. If y, z are given, we can find x from
the equation. The image of the column vector $v = A^T$ is the line spanned by this
vector. This line is perpendicular to the plane and illustrates the fundamental theorem
of linear algebra assuring the kernel of A being perpendicular to the image of A^T or
equivalently the kernel of A^T being perpendicular to the image of A.

7.2. **Curves** are objects of dimension 1. For example, the line spanned by a vector v is written as the set of points $r(t) = tv = [t, 2t, 3t]^T$. We call this a **parametrization** of the line. The free variable t is called **time**. It determines, where we are located at a fixed time t. At time $t = 12$ for example we are positioned at the point $(12, 24, 36)$ corresponding to the vector $[12, 24, 36]^T$. [1] The vector v has the interpretation of a **velocity**. It tells us how fast we move on the line. Of course, replacing v with $3v$ would give us the same line but we would travel three times faster and would reach the point $(12, 24, 36)$ three times faster.

7.3. If the velocity can change direction and length, we can drive around on more interesting paths. The frame work is to take three continuous functions $x(t), y(t), z(t)$ and look at the path $(x(t), y(t), z(t))$ in space. We write this in vector notation as $r(t) = [x(t), y(t), z(t)]^T$. Now, because we get tired always to write the T pointing out that we use column vectors, we will just write $r(t) = [x(t), y(t), z(t)]$. Most of the time, we assume the functions to be differentiable but the case of a ping-pong ball bouncing off a table shows that also non-smooth curves can matter even in daily life. Curves can be very complicated. Take a ping-pong ball and place it into an elliptic container. The billiard path it traces is chaotic. In this lecture we look at curves given by parameterizations, learn how to take derivative to get the velocity or acceleration. We also learn how to integrate. This allows us to compute paths. We can for example compute where a ball falling in a gravitational field is at time t.

LECTURE

7.4. Given n continuous functions $x_j(t)$ of one variable t, we can look at the vector-valued function $r(t) = [x_1(t), \ldots, x_n(t)]^T$. We call it a **parametrized curve**. An example is $r(t) = [3 + 2t, 4 + 6t]$ which is a line through the point $(3, 4)$ and containing the vector $[2, 6]$. [2] If t is in the **parameter interval** $a \leq t \leq b$, then the image of r is $\{r(t) \mid a \leq t \leq b\}$, which defines a **curve** in $\mathbb{R}^n$. The curve **starts** at the point $r(a)$ and **ends** at the point $r(b)$. An other important example is the **circle** $r(t) = [\cos(t), \sin(t)]$, where t is in the interval $[0, 2\pi]$. Its image is a circle in the plane $\mathbb{R}^2$. The parametrization $r(t)$ contains more information than the curve itself: the parabolic curve $r(t) = [t, t^2]$ defined on $t \in [-1, 1]$ for example is the same as the curve $r(t) = [t^3, t^6]$ for $\in [-1, 1]$, but in the second parametrization, the curve is traveled with different speed. Curves in $\mathbb{R}^3$ can be admired in our physical space like $r(t) = [x(t), y(t), z(t)] = [t\cos(t), t\sin(t), t]$ which is a spiral. This particular curve is contained in the cone $x^2 + y^2 = z^2$.

7.5. If the functions $t \to x_j(t)$ are differentiable, we can form the derivative $r'(t) = [x_1'(t), \ldots, x_n'(t)]$. While this technically is again a curve, we think of $r'(t)$ as a vector attached to the point $r(t)$ and say that $r'(t)$ is **tangent** to $r(t)$. The length $|r'(t)|$ of the velocity is called the **speed** of r. If also higher derivatives of the functions $x_j(t)$ exist, we can form the second derivative $r''(t)$ called the **acceleration** or third derivative $r'''(t) = r^{(3)}(t)$ called the **jerk**. Then come **snap** $r^{(4)}(t)$, **crackle** $r^{(5)}(t)$ and **pop** $r^{(6)}(t)$ and the **Harvard** $r^{(7)}(t)$ introduced in the fall of 2016 in a multi-variable exam.

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[1] We can associate any vector v with a point. Think of the vector as connecting 0 with the point.

[2] To reduce clutter, we write row vectors $[2, 6, 1]$ rather than column vectors $[2, 6, 1]^T$

7.6. Given the first derivative function $r'(t)$ as well as the initial point $r(0)$, we can get back the function $r(t)$ thanks to the **fundamental theorem of calculus**. Because of **Newton's law** which tells that a mass point of mass m subject to a force field F depending on position and velocity satisfies the **Newtonian differential equation** $mr''(t) = F(r(t), r'(t))$, the following result is important:

Theorem: $r(t)$ is uniquely determined from $r''(t)$ and $r(0)$ and $r'(0)$.

Proof. In each coordinate we get $x'_k(t) = \int_0^t x''_k(s)\, ds + x'_k(0)$ and $x_k(t) = \int_0^t x'_k(s)\, ds + x_k(0)$. We have just applied twice the **fundamental theorem of calculus**. $\square$

A special case is if $r''(t)$ is constant. A special case is the **free fall situation**. The coordinate functions are then quadratic. Assume $r''(t) = [0, 0, -10]$, and $r'(0) = [0, 0, 0]$ and $r(0) = [0, 0, 20]$, then $r(t) = [0, 0, 20 - 5t^2]$. If you jump from 20 meters into a pool, you need $t = 2$ seconds to hit the water.

7.7. Given a curve $r(t)$ for which the velocity $r'(t)$ is never zero, we can form the **unit tangent vector** $T(t) = r'(t)/|r'(t)|$. If $T'(t)$ is never zero, we can then form $N(t) = T'(t)/|T'(t)|$, the **normal vector**. The vector $B = T \times N$ is called the **binormal vector**. The scalar $|T'(t)|/|r'(t)|$ is called the **curvature** of the curve.

Theorem: In $\mathbb{R}^3$, we have $K = |T'|/|r'| = |r' \times r''|/|r'|^3$.

Proof. We will do this computation in class. $\square$

7.8. Even if $r(t)$ is perfectly smooth, the curvature can become infinite. Lets look at the example $r(t) = [t^2, t^3, 0]$. Then $r'(t) = [2t, 3t^2, 0]$ and $r''(t) = [2, 6t, 0]$ and $r'(t) \times r''(t) = [0, 0, 6t^2]$. The curvature is $(6/t)(4 + 9t^2)^{-3/2}$ which has a singularity at $t = 0$.

7.9. Even when $r(t)$ is perfectly smooth and never zero, the normal vector can depend in a discontinuous way on t. Example: $r(t) = [t, t^3/3]$. Now $r'[t] = [1, t^2]$ and $T(t) = [0, t^2]/\sqrt{1 + t^4}$. We see that $T'(t)$ takes different signs in the second coordinate. After normalization we have $\lim_{t \to 0, t>0} N(t) = [0, 1]$ and $\lim_{t \to 0, t<0} N(t) = [0, -1]$. At the **inflection point** of the graph of the cube function, the concavity has changed from concave down to concave up. This has changed the direction of the normal vector N.

7.10. Side remark. We have looked at parametrized vectors only. If the entries $A_{ij}(t)$ of a matrix depend on times we have a matrix valued curve $A(t)$. This appears in differential equations, in quantum mechanics (operators moving in time) or - most importantly - in moving pictures! A movie is just a matrix valued curve.

7.11. Side remark. A planar curve $r(t) = [x(t), y(t)]^T$ in the plane defined on $t \in [0, 2\pi]$ is called a **simple closed curve** if $r(0) = r(2\pi)$ and there are no values $0 \le s \ne t < 2\pi$ for which $r(t) = r(s)$. For a smooth curve, meaning that the first two derivatives exist, we can look at the polar angle $\alpha(t)$ of the vector $r'(t)$. Define the **signed curvature** of the curve as $\kappa(t) = \alpha'(t)/|r'(t)|$. We have $|\kappa(t)| = K(t)$. The **Hopf Umlaufsatz** tells $\int_0^{2\pi} \kappa(t)\, dt = 2\pi$. In the case of the circle for example, $\kappa(t) = 1$.

7.12. Side remark. We can verify that any curve $r(t)$ parametrized on $[a, b]$ such that $r'(t) \neq 0$ for all $t \in [a, b]$ can be parametrized as $R(t)$ on $[a, b]$ such that $|R'(t)| = 1$ for all t. Proof: we look for a monotone function $s(t)$ such that the derivative of $r(s(t))$ has length 1. This means we want $|r'(s(t))|s'(t) = 1$. In other words, look for a function $s(t)$ such that $s'(t) = 1/|r'(s(t))| = F(s(t))$ and $s(a) = 0$. This is what we call a differential equation. There is a general existence theorem for differential equations (proven later) which assures that there exists a unique solution $s(t)$. End of proof. The result is very intuitive. You can drive from $r(a)$ to $r(b)$ along the curve traced by $r(t)$ by just keeping the speed 1. This gives your your new parametrization. Your new time interval will be $[0, L]$ where L is the arc length (the length of your trip). We will come to arc length computation in the next lesson.

7.13. Side remark. Continuous curves can be complicated: If you look at the pollen particle in a microscope, it moves erratically on a curve which is nowhere differentiable as it is constantly bombarded with air molecules which bounce it around. This is **Brownian motion**. There are also **Peano curves** or **Hilbert curves** $[0, 1] \to [0, 1]^2$ or space filling Hilbert curves $r(t) : [0, 1] \to Q = [0, 1]^3$ which cover every point of the **cube** Q. These curves define a continuous bijection from $[0, 1]$ to $[0, 1]^3$. (The inverse is not continuous. Still, the construction shows that there are the same number of points in $[0, 1]$ than in $[0, 1]^3$).
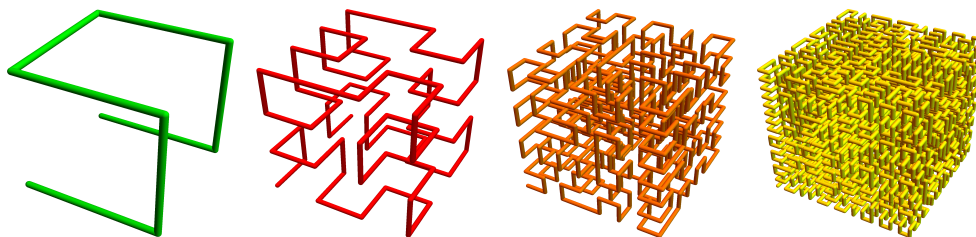

FIGURE 2.   The four first stages in the construction of a space filling curve.

EXAMPLES

7.14. Assuming the **Newton equations** $mr''(t) = F(t)$, find the path $r(t)$ of a body of mass $m = 1/2$ subject to a force $F(t) = [\sin(t), \cos(t), -10]$ with $r(0) = [3, 4, 5]$ and $r'(0) = [1, 2, 7]$. Solution: we have $r''(t) = [2\sin(t), 2\cos(t), -20]$. Integration gives $r'(t) = [-2\cos(t), 2\sin(t), -20t] + [c_1, c_2, c_3]$. Fixing the constants gives $r'(t) = [3 - 2\cos(t), 2 + 2\sin(t), 7 - 20t]$. A second integration gives $r(t) = [3t - 2\sin(t), 2t - 2\cos(t), 7t - 10t^2] + [c_1, c_2, c_3]$ with other constants $C = [c_1, c_2, c_3]$. Comparing $r(0) = [0, -2, 0] + [c_1, c_2, c_3] = [3, 4, 5]$ gives $r(t) = [3 + 3t - 2\sin(t), 6 + 2t - 2\cos(t), 5 + 7t - 10t^2]$.

7.15. Let $r(t) = [L\cos(t), L\sin(t), 0]$. Then $r'(t) = [-L\sin(t), L\cos(t), 0]$ and $r''(t) = [-L\cos(t), -L\cos(t), 0]$ and $r'(t) \times r''(t) = [0, 0, L^2]$ and $|r'(t)| = L$ so that $|r'(t) \times r''(t)|/|r'(t)|^3 = 1/L$. A circle of radius L has curvature $1/L$!

7.16. A closed simple curve C in $\mathbb{R}^3$ is a **knot**. For any positive integer n, m we can look at the **torus knot** $r(t) = [(3 + \cos(mt))\cos(nt), (3 + \cos(mt))\sin(nt), \sin(mt)]$. The **total curvature** of a knot is defined as $\int_0^{2\pi} K(t)\, dt$. See Figure 3. [3]

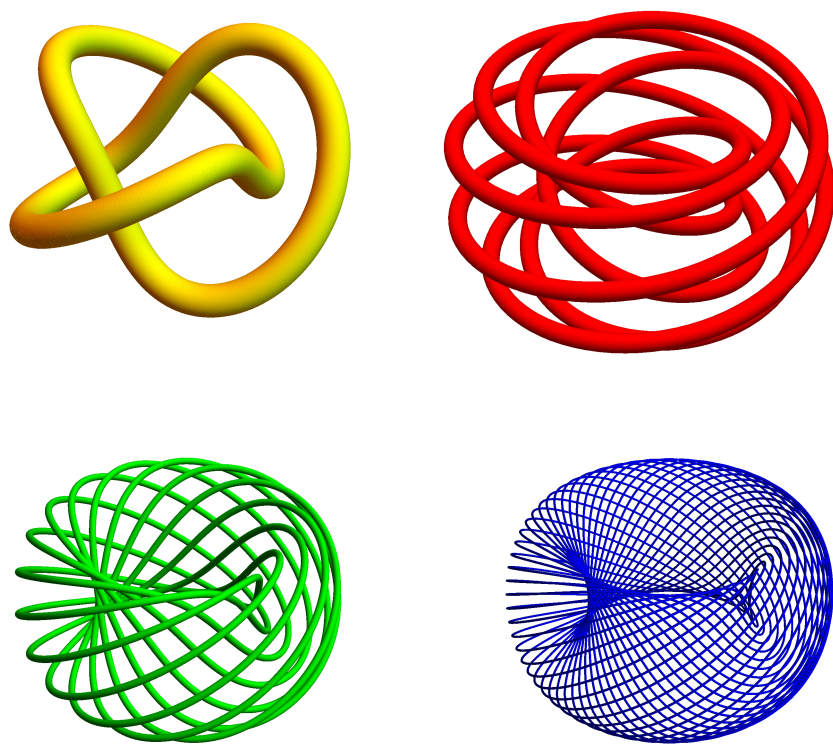[3]A general theorem of Fay and Milnor assures that a knot of total curvature $\leq 4\pi$ is trivial.

FIGURE 3. Torus knots $T(2, 3), T(7, 3), T(12, 13)$ and $T(30, 43)$. Their total curvatures are $38.6, 245.6, 487.2, 2167.3$.
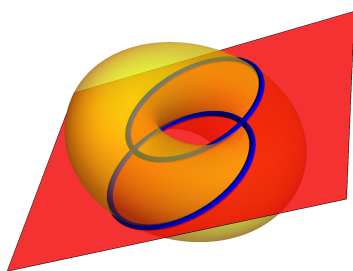

FIGURE 4. Villarceau circles obtained when slicing a bagel. Given a two surfaces, it can be tough to find the intersection.

Homework

Problem 7.1: You sit on a bench at $A = r(0) = [0, 0, 3]$ near the frozen Charles located between Winthrop and Elliot and chip stones aiming at $B = [0, -300, 15]$, a point near the Harvard business school. In order not to get into trouble, we assume everything happens in our imagination and that the stone is friction-free. You use a sling shot and throw with initial velocity $r'(0) = [0, -24, 61]$, assume the gravitational acceleration to be $r''(t) = [0, 0, -10]$ at all times and use meters for distance and seconds for time. At which point does the stone reach the 15 meter height mark while descending? [Optional: you like a challenge and want to bounce off on the ice surface at $C = [0, -200, 0]$ and reach the point B. What initial velocity v at A does achieve this?]

Problem 7.2: We want to produce a logo for a new company and experiment. Draw the curve $r(t) = [\cos(t), \sin(t)] + [\cos(11t), \sin(9t)]/4$ and find the velocity, acceleration, and curvature at $t = 0$.

Problem 7.3: Parametrize the curve $r(t)$ obtained by intersecting the cylinder $x^2/9 + y^2/4 = 1$ with the plane $z = x + 5y$.

Problem 7.4: Verify that the **torus knot** $r(t) = [x(t), y(t), z(t)] = [(2+\cos(mt))\cos(nt), (2+\cos(mt))\sin(nt), \sin(mt)]$ lives on the torus $(3 + x^2 + y^2 + z^2)^2 - 16(x^2 + y^2) = 0$.

Problem 7.5: You slice a bagel in a non-standard way. Let us assume that the bagel is given by $(x^2 + y^2 + z^2 + 16)^2 - 100(x^2 + y^2) = 0$. Verify that if we intersect this torus with the plane $3x = 4z$, then we get the **Villarceau circles** $r(t) = [4\cos(t), 3 + 5\sin(t), 3\cos(t)]$ as well as the circle $r(t) = [4\cos(t), -3 + 5\sin(t), 3\cos(t)]$.

FIGURE 5. The scenery for our imaginary dream sling shot experiment.