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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 28: Second Hourly (Practice A)

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 28.2 and 28.3, we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

Archimedes sends his good luck wishes. He unfortunately can not join us as he is "busy proving a new theorem". He just sent us his selfie. Oh well, these celebrities!

Problems

Problem 28A.1 (10 points):

a) (4 points) Prove that if x^3 is irrational, then x is irrational.

b) (3 points) Prove or disprove: the product of two odd integers is odd.

c) (3 points) Prove or disprove: the sum of two odd integers is odd.

Solution:

a) Use contradiction. Assume x is rational. Then $x = p/q$ and so $x^3 = p^3/q^3$ is rational too. This contradicts that x is irrational.

b) If $x = 2k + 1, y = 2l + 1$ are odd, then $xy = (4kl + 2k + 2l + 1) = 2m + 1$ is odd with $m = 2kl + k + l$. The direct proof is better than the indirect.

c) This is false. 1 is odd, 3 is odd but $1 + 3$ is even.

Problem 28A.2 (10 points) Each question is one point:

a) What is the name of the partial differential equation $f_{tt} = f_{xx}$?

b) The series $f(x) = \sum_{k=0}^{\infty} x^k/k! = 1 + x + x^2/2! + x^3/3! + \cdots$ represents a function. Which one?

c) The implicit differentiation formula for $f(x, y(x)) = 1$ is $y'(x) = \dots\dots$

d) What is the name of the function $f(s) = \sum_{n=1}^{\infty} n^{-s}$?

e) On a circular island there are exactly 3 maxima and one minimum for the height f. Assuming f is a Morse function, how many saddle points are there?

f) Which mathematician first found the value for the volume of the ball $x^2 + y^2 + z^2 \leq 1$?

g) True or False: the directional derivative of f in the direction $\nabla f(x)/|\nabla f(x)|$ is negative at a point where ∇f is not zero.

h) The equation $f(x + t) = e^{Dt} f = f(x) + f'(x)t + f''(x)t^2/2 + \cdots$ solves a partial differential equation. Which one?

i) What is the formula for the surface area of a surface S parametrized by $r(u, v)$ over a domain R?

j) What is the integration factor (= distortion factor) when going to spherical coordinates (ρ, ϕ, θ)?

Solution:
a) Wave,
b) e^x
c) $y' = -f_x/f_y$.
d) Zeta function.
e) 3
f) Archimedes.
g) False. Functions dance upwards.
h) transport equation
i) $\int\int_R |r_u \times r_v| \, dudv$
j) $\rho^2 \sin(\phi$.

Problem 28A.3 (10 points) Each question is two points:

We see the level curves of a Morse function f. The circle through ABC will sometimes serve as a constraint $g(x,y) = x^2 + y^2 = 1$. In all questions, we only pick points from A,B,C,D,E,F,G,H,I,J,K,L,M.

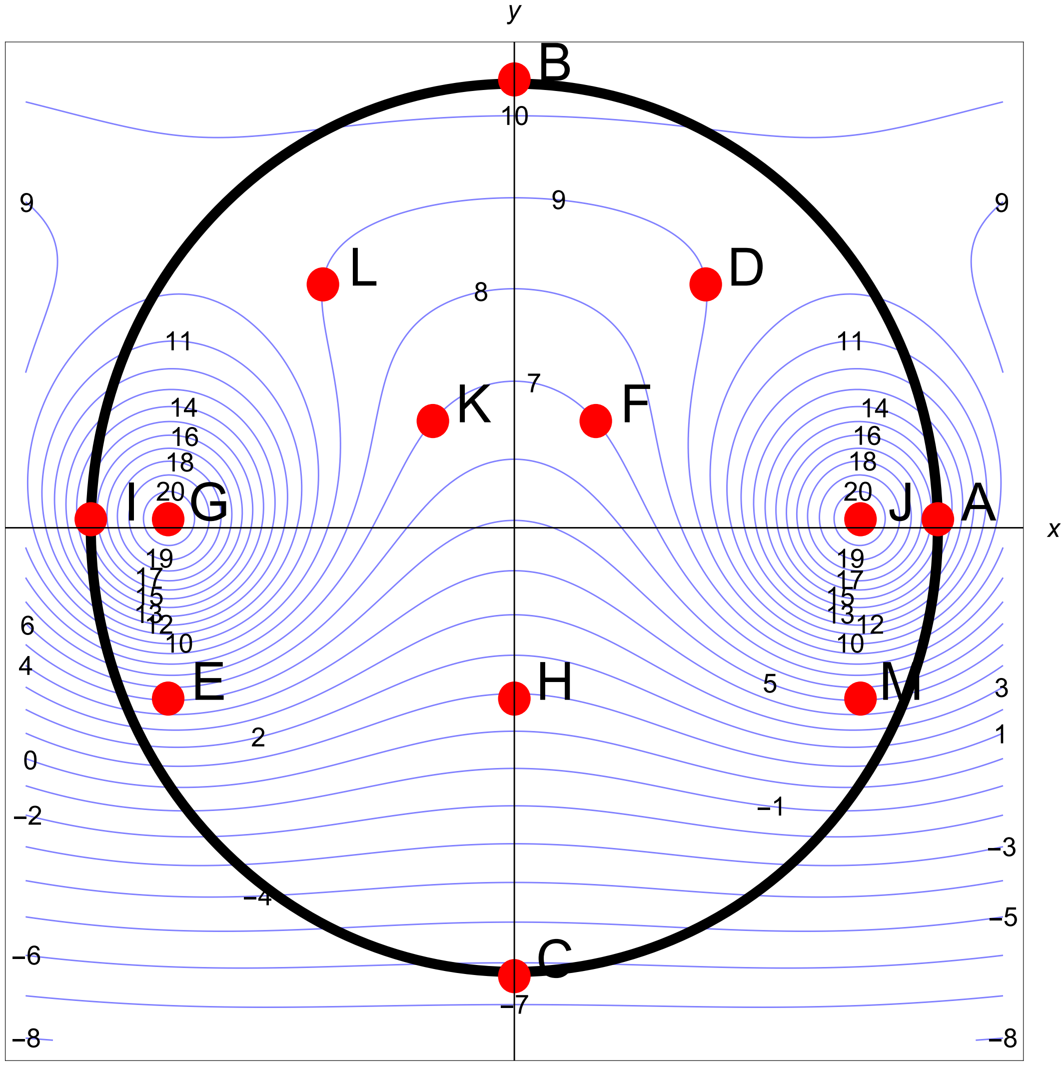
a) Which points are local minima of f under the constraint $g(x,y) = 1$.

b) Which points are local maxima of f under the constraint $g(x,y) = 1$.

c) At which points do we have $f_x(x,y) \cdot f_y(x,y) \neq 0$?

d) At which points are $|\nabla f(x,y)|$ maximal?

e) At which points are $|\nabla f(x,y)|$ minimal?



Solution:
This was a slaughter house. Only two students got everything right!
a) C
b) BAI
c) KF
d) IA
e) GJ

Problem 28A.4 (10 points):

a) (5 points) Find the tangent plane to the surface

$$f(x, y, z) = x^2y - x^3 + y^2 + z^4xy = -13$$

at the point $(2, -1, 1)$.

b) (5 points) Estimate $f(2.001, -0.99, 1.1)$ by linear approximation.

Solution:
The gradient of the function at the point is $[-17, 4, 8]^T$.
a) The equation of the tangent plane is $-17x + 4y - 8z = -46$
b) We estimate $-13 - 17 * 0.001 + 4 * 0.01 - 8 * 0.1 = -13.777$.

Problem 28A.5 (10 points):

a) (5 points) Find the quadratic approximation $Q(x, y)$ of

$$f(x, y) = 5 + x + y + x^2 + 3y^2 + \sin(xy) + e^x$$

at $(x, y) = (0, 0)$.

b) (5 points) Estimate the value of $f(0.001, 0.02)$ using quadratic approximation.

Solution:
a) We get $Q(x, y) = 6 + 2x + y + 3x^2/2 + xy + 3y^2$.
b) Evaluate this at the point gives 6.02322.

Problem 28A.6 (10 points):

a) (8 points) Classify the critical points of the function

$$f(x, y) = x^2 - y^3 + 2x + 3y$$

using the second derivative test.

b) (2 points) Does the function $f(x, y)$ have a global minimum or global maximum?

Solution:
a) There is a minimum at $(-1,-1)$ and a saddle at $(-1,1)$. The discriminant at the first point is 12, at the second point -12. f_{xx} is equal to 2 at both points.
b) No global maximum, nor minimum: look at $f(0,y)- = -y^3+3y$ which is unbounded below and above.

Problem 28A.7 (10 points):

Using the Lagrange optimization method, find the parameters (x,y) for which the area of an arch

$$f(x,y) = 2x^2+4xy+3y^2$$

is minimal, while the perimeter

$$g(x,y) = 8x+9y = 33$$

is fixed.

Solution:
First write down the Lagrange equations $\nabla f = \lambda \nabla g$ and eliminate λ.

$$\begin{aligned} 4x+4y &= \lambda 8 \\ 4x+6y &= \lambda 9 \\ 8x+9y &= 33\,. \end{aligned}$$

We end up with $(4x+4y)9 = 8(4x+6y)\lambda$. Eliminating λ gives $y = x/3$. Plugging this into the constraint gives $\boxed{x=3, y=1}$.

Problem 28A.8 (10 points):

a) (5 points) Find the moment of inertia

$$I = \int\int_G (x^2+y^2)\;dydx$$

of the quarter disc $G = \{x^2+y^2 \leq 1, x \geq 0, y \leq 0\,\}$.

b) (5 points) Evaluate the double integral

$$\int_1^e \int_{\log(x)}^1 \frac{y}{e^y-1}\;dydx\;,$$

where log is the natural log as usual.

Solution:
a) Use polar coordinates! For right half disc, we have

$$\int_0^1 \int_{-\pi/2}^0 r^3 \; d\theta dr = \pi/8 \; .$$

We wanted to see the right integral too.
The result is $\boxed{\pi/4}$.

b) Change the order of integration! It is important here to make a picture! We end up with the integral

$$\int_0^1 \int_1^{e^y} \frac{y}{e^y - 1} \; dx dy \; .$$

Now, the inner integral is no problem and gives y. The result is $\int_0^1 y \; dy = \boxed{1/2}$.

Problem 28A.9 (10 points):

Find the integral

$$\iiint_E f(x, y, z) \; dz dy dx$$

of the function

$$f(x, y, z) = x + (x^2 + y^2 + z^2)^4$$

over the solid

$$E = \{(x, y, z) \mid 1 \leq x^2 + y^2 + z^2 \leq 4, \quad z \geq 0 \} \; .$$

Solution:
Use spherical coordinates: $\int_0^{2\pi} \int_0^{\pi/2} \int_1^2 (\rho \cos(\theta) \sin(\phi) + \rho^8) \rho^2 \sin(\phi) \; d\rho d\phi d\theta = 2\pi(2^11 - 1)/11$.

Problem 28A.10 (10 points):

Find the surface area of

$$r(x, y) = \begin{bmatrix} 2x \\ y \\ \frac{x^3}{3} + y \end{bmatrix}$$

with $0 \leq x \leq 2$ and $0 \leq y \leq x^3$.

Solution:
$\int_0^2 \int_0^{x^3} \sqrt{x^4+8}\, dydx$. This gives $(24^{3/2} - 8^{3/2})/6$.