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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 28.2 and 28.3, we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

Archimedes sends his good luck wishes. He unfortunately can not join us as he is "busy proving a new theorem". He just sent us his selfie. Oh well, these celebrities!

Unit 28: Second Hourly (Practice B)

Problems

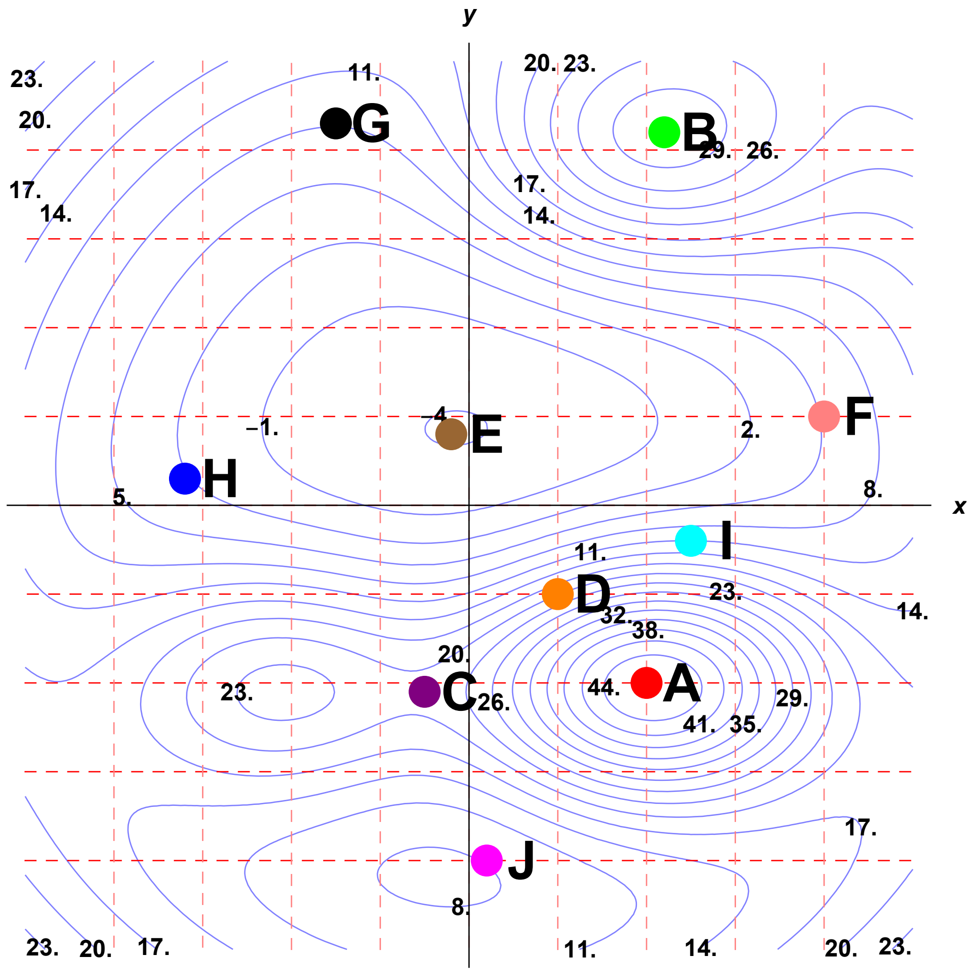
Problem 28B.1 (10 points):
a) (4 points) You know the positive integer n^5 is odd. Prove that n is odd.
b) (3 points) Prove or disprove: if a and b are irrational, then ab is irrational.
c) (3 points) Prove or disprove: if a and b are irrational, then $a+b$ is irrational.

Solution:
a) We prove by $\boxed{\text{contradiction:}}$ if n is even, then $n = 2k$. But then $n^5 = 2^5k^5$ is even. Contradiction.
b) Take $a = b = \sqrt{2}$. $\boxed{\text{The statement is false.}}$
c) Take $a = -b = \sqrt{2}$. $\boxed{\text{The statement is false.}}$

Problem 28B.2 (10 points, each sub problem is one point):
a) What is the name of the differential equation $f_t = f_{xx}$?
b) What assumptions need to hold so that $f_{xy} = f_{yx}$ is true?
c) The gradient $\nabla f(x_0)$ has a relation to $f(x) = c$ with $c = f(x_0)$. Which one?
d) The linear approximation of f at x_0 is $L(x) = f(x_0) + \ldots..$ Complete the formula.
e) Assume f has a maximum on $g = c$, then either $\nabla f = \lambda \nabla g, g = c$ holds or ...
f) Which mathematician proved the switch the order of integration formula?
g) True or false: the gradient vector $\nabla f(x)$ is the same as $df(x)$.
h) The equation $u_t + uu_x = u_{xx}$ is an example of a differential equation. We have seen two major types (each a three capital letter acronym). Which type is it?
i) What is the formula for the arc length of a curve C?
j) What is the integration factor $|d\phi|$ when going into polar coordinates?

Solution:
a) The heat equation.
b) The function needs to have second derivatives which are continuous.
c) The gradient is perpendicular to the level curves or level surfaces.
d) $L(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$
e) or $\nabla g = 0$.
f) Fubini.
g) False. The gradient is a column vector, df is a row vector.
h) It is a PDE.
i) $\int_a^b |r'(t)| \, dt$.
j) r.

Problem 28B.3 (10 points, 2 points for each sub-problem):
We see the level curves of a Morse function f. Only pick points A-J.
a) Which point is critical with discriminant $D = \det(d^2 f) < 0$.
b) At which point is $f_x > 0, f_y = 0$?
c) At which point is $f_x > 0, f_y > 0$?
d) Which (x_0, y_0) are critical points of f when imposing the constraint $g(x, y) = y = y_0$?
e) Which (x_0, y_0) are critical points of f when imposing the constraint $g(x, y) = x = x_0$?



Solution:
a) C
b) F
c) J
d) A,B,C,E,G,I
e) A,B,C,E,F

Problem 28B.4 (10 points):
a) (5 points) Find the tangent plane to the surface $xyz + x^5y + z = 11$ at $(1, 2, 3)$.
b) (5 points) Near $(x, y) = (1, 2)$, we can write $z = g(x, y)$. Find $g_x(1, 2), g_y(1, 2)$.

Solution:
a) $\nabla f = [yz + 5x^4 y, xz + x^5, 1 + xy]$ $\nabla f(1, 2, 3) = [16, 4, 3]$.
The equation is $\boxed{16x + 4y + 3z = 33}$.
b) $g_x = -f_x/f_z = -16/3, g_y = -f_y/f_z = -4/3$.

Problem 28B.5 (10 points):
a) Find the quadratic approximation of $f(x, y, z) = 1 + x + y^2 + z^3 + \sin(xyz)$ at $(0, 0, 0)$.
b) Estimate $f(0.01, 0.03, 0.05)$ using linear approximation.

Solution:
a) We have $df(x, y, z) = [1 + yz\cos(xyz), 2y + xz\cos(xyz), 3z^2 + xy\cos(xyz)]$. $\boxed{Q(x, y, z) = 1 + x + y^2}$. All other terms are zero.
b) Evaluate $L(x, y, z) = 1 + x$ at $(0.01, 0.03, 0.05)$. This is $1 + 0.01 = \boxed{1.01}$.

Problem 28B.6 (10 points):
a) (8 points) Classify the critical points of the function $f(x, y) = x^{12} + 12x^2 + y^{12} + 12y^2$ using the second derivative test.
b) (2 points) Does f have a global minimum? Does f have a global maximum?

Solution:
a) To get the critical points, look where $\nabla f(x, y) = [12x^1 1 + 24x, 12y^1 1 + 24y]$ is $[0, 0]$. This is at $x = 0, x^1 0 = -1$ or $y = 0, y^1 0 = -1$. he only critical point is $(0, 0)$. The discriminant is $D = 24^2 > 0$. Since $f_{xx} > 0$, $\boxed{\text{the point } (0,0) \text{ is a } \textbf{minimum}}$.
b) $\boxed{\text{There is a global minimum}}$: the point $(0, 0)$ because the function is positive everywhere else. There is no global maximum.

Problem 28B.7 (10 points):
On the top of a MIT building there is a radar dome in the form of a spherical cap. Insiders call it the **"Death star" radar dome**. We know that with the height h and base radius r, we have volume and surface area given by $V = \pi r h^2 - \pi h^3/3$, $A = 2\pi r h = \pi$. This leads to the problem to extremize

$$f(x,y) = xy^2 - \frac{y^3}{3}$$

under the constraint

$$g(x,y) = 2xy = 1 \ .$$

Find the minimum of f on this constraint using the Lagrange method!

Solution:
The Lagrange equations lead to $x = y$ and so $\boxed{x = y = 1/\sqrt{2}}$.

Problem 28B.8 (10 points):
Find

$$\iint_R 5/(x^2 + y^2) \; dxdy \ ,$$

where R is the region $1 \leq x^2 + y^2 \leq 25$, $y^2 > x^2$.

Solution:
Use polar coordinates $2\int_{\pi/4}^{3\pi/4} \int_1^5 5r/r^2 \; dr d\theta = 2\pi/25 \log(5) =$ which is $\boxed{5\pi \log(5)}$.

Problem 28B.9 (10 points):
Integrate $f(x,y,z) = z$ over the solid E bound by

$$z = 0$$

$$x = 0$$

$$y = 0$$

and

$$x + y + z = 1 \ .$$

Solution:
The solid has as a base the triangle R with vertices $(0,0),(1,0),(0,1)$. The roof function is $1-x-y$. We have

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dzdydx = \int_0^1 \int_0^{1-x} (1-x-y)^2/2 \, dydx = \frac{1}{24} \; .$$

The result is $\boxed{1/24}$.

Problem 28B.10 (10 points):
What is the surface area of the surface

$$r(x,y) = \begin{bmatrix} 2y \\ x \\ \frac{y^3}{3} + x \end{bmatrix}$$

with $0 \leq y \leq 2$ and $0 \leq x \leq y^3$?

Solution:
Set up the integral. First compute $r_x \times r_y$, then its length $\sqrt{y^4+8}$, then integrate:

$$\int_0^2 \int_0^{y^3} \sqrt{y^4+8} \, dxdy \; .$$

The result is $\boxed{(24^{3/2} - 8^{3/2})/6}$.