LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 28: Keywords for Second Hourly

This is a bit of a checklist. Make your own list. But here is a checklist which tries to be comprehensive. Check off the topics you know and check back with things you do not recall. You will need to have the following on your finger tips.

Partial Derivatives

- $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ partial derivative
- $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ linear approximation
- $Q(x,y) = L(x_0,y_0) + f_{xx}(x-x_0)^2/2 + f_{yy}(y-y_0)^2/2 + f_{xy}(x-x_0)(y-y_0)$ quadratic
- $L(x,y)$ estimates $f(x,y)$ near $f(x_0,y_0)$. The result is $f(x_0,y_0) + a(x-x_0) + b(y-y_0)$
- tangent line: $ax + by = d$ with $a = f_x(x_0,y_0), b = f_y(x_0,y_0)$, $d = ax_0 + by_0$
- tangent plane: $ax + by + cz = d$ with $a = f_x, b = f_y, c = f_z$, $d = ax_0 + by_0 + cz_0$
- estimate $f(x,y,z)$ by $L(x,y,z)$ near (x_0,y_0,z_0)
- $f_{xy} = f_{yx}$ Clairaut's theorem, if f_{xy} and f_{yx} are continuous.
- $r_u(u,v)$, $r_v(u,v)$ tangent to surface parameterized by $r(u,v)$

Partial Differential Equations

- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ wave equation
- $f_x - f_t = 0$ transport equation
- $f_{xx} + f_{yy} = 0$ Laplace equation
- $f_t + f f_x = f_{xx}$ Burgers equation
- $f_x^2 + f_y^2 = 1$ Eiconal equation
- $f_t = f - x f_x - x^2 f_{xx}$ Black Scholes

Gradient

- $\nabla f(x,y) = df^T = [f_x, f_y]^T$, $\nabla f(x,y,z) = [f_x, f_y, f_z]^T$, gradient
- $D_v f = \nabla f \cdot v$ directional derivative
- $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r\,'(t)$ chain rule
- $\nabla f(x_0,y_0)$ is orthogonal to the level curve $f(x,y) = c$ containing (x_0,y_0)
- $\nabla f(x_0,y_0,z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ containing (x_0,y_0,z_0)
- $\frac{d}{dt} f(x + tv) = D_v f$ by chain rule
- $(x-x_0)f_x(x_0,y_0,z_0) + (y-y_0)f_y(x_0,y_0,z_0) + (z-z_0)f_z(x_0,y_0,z_0) = 0$ tangent plane
- $f(x,y)$ increases in the $\nabla f/|\nabla f|$ direction. Functions dance upwards.
- $f(x,y,z) = c$ defines $z = g(x,y)$, and $g_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$ implicit diff

Extrema

$\nabla f(x,y) = [0,0]^T$, critical point or stationary point
$D = f_{xx}f_{yy} - f_{xy}^2 = \det(df)$ discriminant, useful in second derivative test
$f(x_0,y_0) \geq f(x,y)$ in a neighborhood of (x_0,y_0) local maximum
$f(x_0,y_0) \leq f(x,y)$ in a neighborhood of (x_0,y_0) local minimum
$\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, or $\nabla g = 0$ Lagrange equations
second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle point**
$f(x_0,y_0) \geq f(x,y)$ everywhere, global maximum
$f(x_0,y_0) \leq f(x,y)$ everywhere, global minimum
f is Morse if the Hessian $H = d^2 f$ is invertible at every critical point

Double Integrals

$\int\int_R f(x,y)\ dydx$ double integral
$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx$ bottom-to-top region
$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy$ left-to-right region
$\int\int_R f(r,\theta) \boxed{r}\ drd\theta$ polar coordinates
$\int\int_R |r_u \times r_v|\ dudv$ surface area
$\int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$ Fubini
$\int\int_R \boxed{1}\ dxdy$ area of region R
$\int\int_R f(x,y)\ dxdy$ signed volume of solid bound by graph of f and xy-plane

Triple Integrals

$\int\int\int_R f(x,y,z)\ dzdydx$ triple integral
$\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx$ integral over rectangular box
$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\ dzdydx$ type I region
$\int\int\int_R f(r,\theta,z)\ \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates
$\int\int\int_R f(\rho,\theta,\phi)\ \boxed{\rho^2 \sin(\phi)}\ d\rho d\phi d\theta$ integral in spherical coordinates
$\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\ dxdydz$ Fubini
$V = \int\int\int_E \boxed{1}\ dzdydx$ volume of solid E
$M = \int\int\int_E f(x,y,z)\ dzdydx$ mass of solid E with density f.

General advise

Draw the region when integrating in in higher dimensions.
Consider other coordinate systems if the integral does not work.
Consider changing the order of integration if the integral does not work.
For tangent planes, compute the gradient $[a,b,c]^T$ first then fix the constant.
When looking at relief problems, mind the gradient.

Theorems

Clairaut, Taylor, Fubini, Island theorem, Sphere and Ball volumes, Morse theorem, chain rule, gradient theorem, change of variables

People

Clairaut, Fubini, Lagrange, Fermat, Riemann, Archimedes, Hamilton, Euler, Taylor, Morse, Hopf, Tao, Polya, Riemann

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