1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total :

Unit 28: Hourly 2

- You only need this booklet and something to write. Please stow away any other material and electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 28.2 and 28.3, we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is also space on the back of each page.
- If you finish a problem somewhere else, please indicate on the problem page so that we find it.
- You have 75 minutes for this hourly.

Archimedes sends his good luck wishes again. He also sits here so that you can not read the first problem until we all start at noon.

Problems

Problem 28.1 (10 points):

a) (3 points) Prove or disprove that the product of a rational and an irrational number is irrational.

b) (3 points) Prove or disprove that the product of two irrational numbers is irrational.

c) (2 points) Prove or disprove that the product of two numbers of the form $4k-1$ is a number of the form $4k-1$.

d) (2 points) Prove or disprove that the product of two numbers of the form $4k+1$ is a number of the form $4k+1$.

Problem 28.2 (10 points) Each question is one point:

a) What is the name of the partial differential equation $f_t = f f_x$?

b) The series $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$ represents a function. Which one?

c) The implicit differentiation formula for $f(x, y, z(x)) = 1$ is $z_x(x) = \dots\dots\dots$

d) The problem to compute the value of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s = 2$ is called the .... problem.

e) Is it possible that we have a Morse function on the 2-sphere $x^2 + y^2 + z^2 = 1$ has 3 maxima, 1 minimum and 3 saddle points?

f) Who proved that one can change the order of integration on a rectangle? The result is called the .... theorem.

g) You measure progress with a Morse function f in a data space $\mathbb{R}^n$. You are located at a point which is not a critical point. In which direction do you have to change the parameters to make f larger?

h) The function $f(x, t) = \sin(x + t) + \sin(x - t)$ is a solution of of one of the basic partial differential equations. Which one?

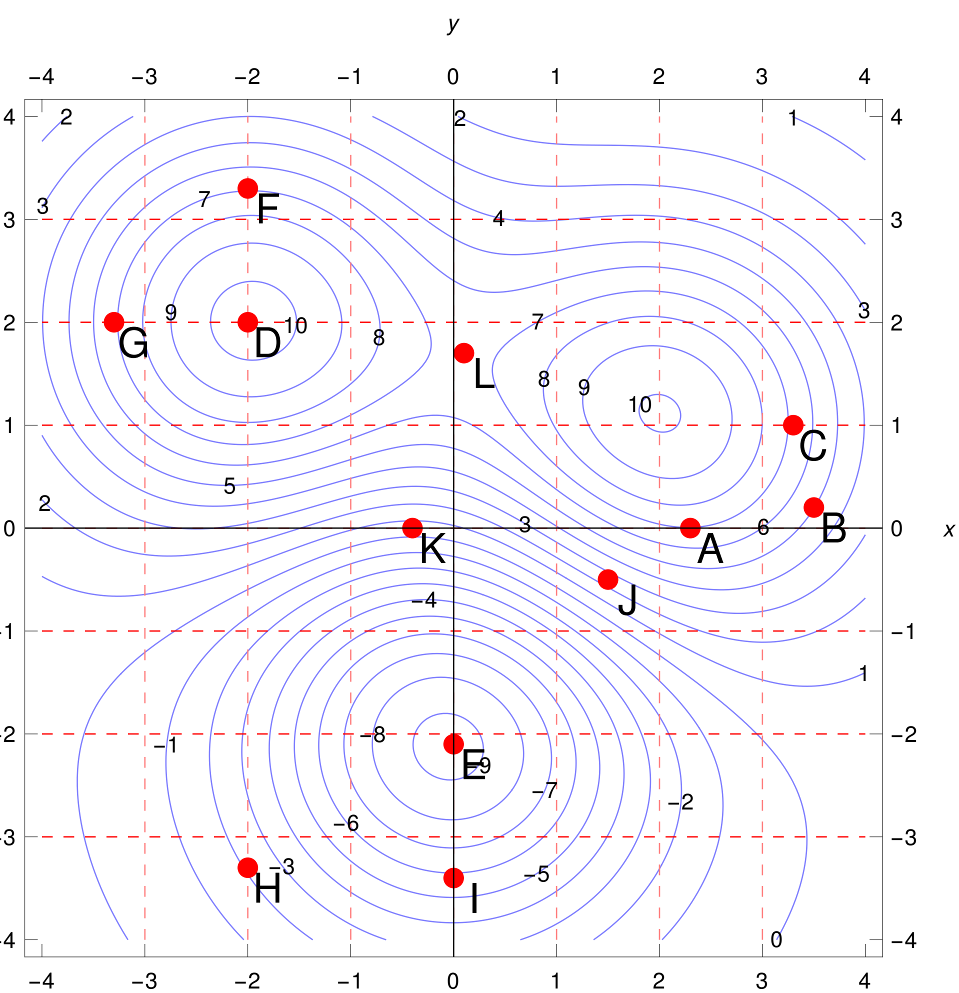
i) What is the distortion factor of the coordinate change $r : \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \to (3x, 4y, 7z)$?

j) You are on Elysium, a torus shaped artificial habitat on which the height function of the hills is a Morse function. There are 5 hills (maxima) and 2 sinks (minima). How many saddle points are there on Elysium?

Problem 28.3 (10 points) Each question is one point:

We see the level curves of a Morse function f. In every of the question, we pick **exactly one point** from A,B,C,D,E,F,G,H,I,J,K,L. Points might appear several times and some points might not appear.

a) Which point is a local maximum?
b) Which point is a local minimum?
c) Which point is a saddle point?
d) Which point is a local minima of f under the constraint $g(x, y) = y = 0$?
e) Which point is a local maxima of f under the constraint $g(x, y) = y = 0$?
f) At which point is $|\nabla f(x, y)|$ maximal among all the points?
g) At which point is $f_x(x, y)$ positive and $f_y(x, y) = 0$?
h) At which point is $f_y(x, y)$ positive and $f_x(x, y) = 0$?
i) At which point are both $f_x(x, y)$ and $f_y(x, y)$ positive?
j) At which point are both $f_x(x, y)$ and $f_y(x, y)$ negative?



Problem 28.4 (10 points):

a) (5 points) Find the tangent hyper plane $ax + by + cz + dw = e$ to the hyper surface

$$f(x, y, z, w) = xy^2z^2 + w = 2$$

at the point $(x_0, y_0, z_0, w_0) = (2, -1, -1, 0)$.

b) (5 points) Estimate $f(2.001, -0.9, -1.01, 0.07)$ by linear approximation.

Problem 28.5 (10 points):

a) (6 points) Classify the critical points of the function

$$f(x, y) = 3 - 3x + x^2 - 3y + xy + y^2$$

using the second derivative test.

b) (2 points) Does the function $f(x, y)$ have a global minimum?

c) (2 points) Does the function $f(x, y)$ have a global maximum?

Problem 28.6 (10 points):

a) (4 points) Find the quadratic approximation $Q(x, y)$ of the function

$$f(x, y) = 3 - 3x + x^2 - 3y + xy + y^2$$

at $(x_0, y_0) = (1, 1)$. We have already seen this function in Problem 28.5).

b) (3 points) Is this function f a Morse function?

c) (3 points) Estimate the value of $f(1.03, 0.2)$ using quadratic approximation.

Problem 28.7 (10 points):

Using the Lagrange optimization method, find the parameters (x, y) for which

$$f(x, y) = 3 - 3x + x^2 - 3y + xy + y^2$$

is maximal or minimal under the constraint

$$g(x, y) = x^2 + y^2 = 2 \; .$$

Problem 28.8 (10 points):

a) (5 points) Evaluate the integral

$$I = \iint_G e^{x^2+y^2} \; dydx$$

of the annular region $G = \{1 \leq x^2 + y^2 \leq 4\}$.

b) (5 points) Evaluate the double integral

$$\int_1^3 \int_0^{9-x^2} \frac{y^2}{\sqrt{9-y}-1} \; dy \; dx \; .$$

Problem 28.9 (10 points):

Integrate

$$\iiint_E (x^2 + y^2 + z^2)^2 \; dzdydx$$

for

$$E = \{(x,y,z) \mid 4 \leq x^2 + y^2 + z^2 \leq 9, \;\; x^2 + y^2 < z^2\} \; .$$

Problem 28.10 (10 points):

Compute the **surface area** of the surface

$$r(u,v) = \begin{bmatrix} 2v\cos(u) \\ 2v\sin(u) \\ u^2 \end{bmatrix}$$

over the region $R = \{u^2 + v^2 \leq 9\}$.

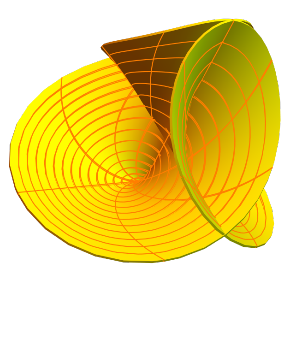


FIGURE 1. The surface in problem 10.

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 22B, HARVARD COLLEGE, SPRING 2022