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Name:

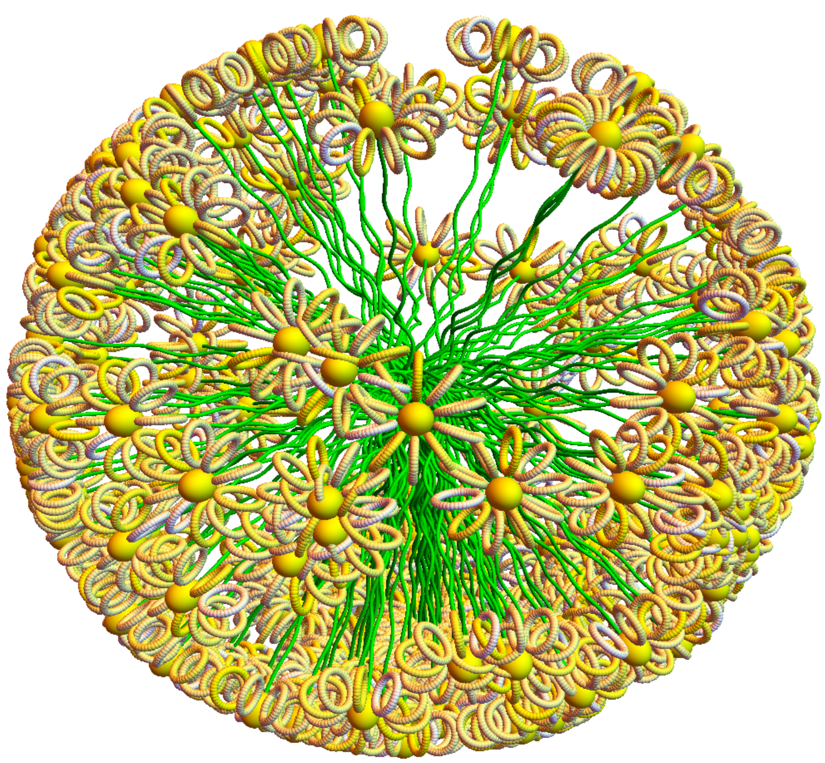
LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Welcome to the final exam. Please don't get started yet. We start all together at 9:00 AM after getting reminded about some formalities. You can fill out the attendance slip already. Also, you can already enter your name into the larger box above.

- You only need this booklet and something to write. Please stow away any other material and any electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 2 and 3 we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is additional space on the back of each page. If you must, use additional scratch paper at the end.
- If you finish a problem somewhere else, please indicate on the problem page where we can find it.
- You have 180 minutes for this 3-hourly.



Unit 39: Final Practice B

Problems

Problem 39B.1) (10 points):
The graph $G = (V, E)$ in Figure 1 represents a discrete surface in which all triangles are oriented counterclockwise. The values of a 1-form = vector field F are given.

a) (2 points) Find the line integral of F along the boundary curve oriented counter clockwise.
b) (2 points) Compute the curl $H = dF$ and write its values into the triangles.
c) (2 points) What is the sum of all curl values? Why does it agree with the result in a)?
d) (2 points) Find also $g = d^*F$ and enter it near the vertices.
e) (1 point) True or False: $\sum_{x \in V} g(x) = 0$.
f) (1 point) True of False: we called $L = dd^*$ the Laplacian of G.

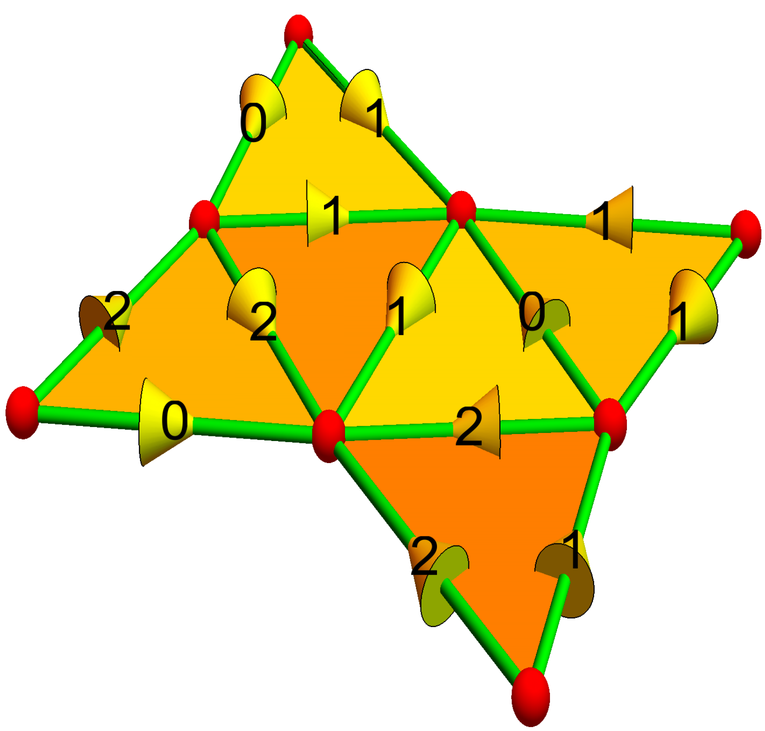
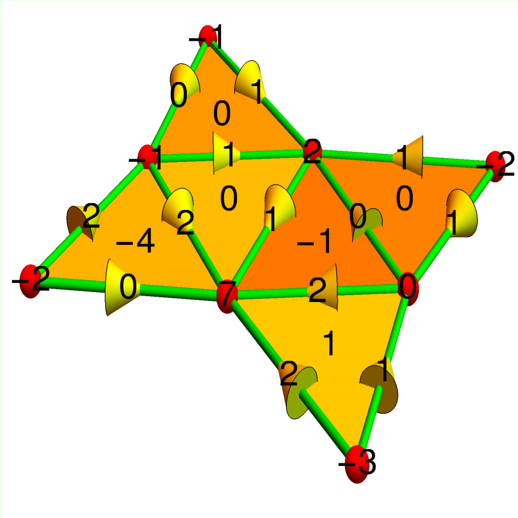


Figure 1. A discrete 2-dimensional region on which a 1-form F models a vector field. You compute the curl dF and divergence d^*F of F.

Solution:
a) -4.
b) See the picture.
The dF and d^*F values are entered. The line integral is -4.
c) The sum is -4
d) see picture
e) True. Always true
f) Not true. We are missing the d^*d.



Problem 39B.2) (10 points) Each question is one point:

a) Name the 3-dimensional analogue of the Mandelbrot set.

b) If A is a 5×4 matrix, then A^T is a $m \times n$ matrix. What is m and n?

c) Write down the general formula for the arc length of a curve

$$r(t) = [x(t), y(t), z(t)]^T$$

with $a \leq t \leq b$.

d) Write down one possible formula for the curvature of a curve

$$r(t) = [x(t), y(t), z(t)]^T \ .$$

e) We have seen a parametrization of the 3-sphere invoking three angles ϕ, θ_1, θ_2. Either write down the parametrization or recall the name of the mathematician after whom it this parametrization is named.

f) The general change of variable formula for $\Phi : R \to G$ is $\iiint_R f(u,v,w) \boxed{} dudvdw = \iiint_G f(x,y,z) \ dxdydz$. Fill in the blank part of the formula.

g) What is the numerical value of $\log(-i)$?

h) We have used the Fubini theorem to prove that C^2 functions $f(x,y)$ satisfy a partial differential equation. Please write down this important partial differential equation as well as its name. (It was used much later in the course.)

i) What is the integration factor $|dr|$ for the parametrization

$$r(u,v) = [a\cos(u)\sin(v), b\sin(u)\sin(v), c\cos(v)]^T \ ?$$

j) In the first lecture, we have defined $\sqrt{\text{tr}(A^T A)}$ as the length of a matrix. What is the length of the 3×3 matrix which contains 1 everywhere?

Solution:
a) Mandelbar set.
b) m=4, n=5.
c) $\int_a^b |r'(t)| dt$
d) $|T'(t)|/|r'(t)|$.
e) Hopf parametrization
f) $|d\Phi|$.
g) $3\pi i/2$.
h) $f_{xy} = f_{yx}$ Clairaut.
i) $A = |r_u \times r_v|$. It does not simplify to something nice as in the case a=b=c.
j) 3.

Problem 39B.3) (10 points) Each problem is 1 point:

a) Assume that for a Morse function $f(x, y)$ the discriminant D at a critical point (x_0, y_0) is positive and that $f_{yy}(x_0, y_0) < 0$. What can you say about $f_{xx}(x_0, y_0)$?

b) We have proven the identity $|dr| = |r_u \times r_v|$, where r was a map from $\mathbb{R}^m$ to $\mathbb{R}^n$. For which m and n was this identity defined?

c) Which of the following is the correct integration factor when using spherical coordinates in 4 dimensions?
$|d\Phi| = r$
$|d\Phi| = (3 + \cos(\phi))$
$|d\Phi| = \rho^2 \sin(\phi)$
$|d\Phi| = \rho^3 \sin(2\phi)/2$

d) Which of the following vector fields are gradient fields? (It could be none, one, two, three or all.)
$F = [x, 0]^T$
$F = [0, x]^T$
$F = [x, y]^T$
$F = [y, x]^T$

e) Which of the following four surfaces is a one-sheeted hyperboloid? (It could be none, one, two, three or all.)
$x^2 + y^2 = z^2 - 1$
$x^2 - y^2 = 1 - z^2$
$x^2 + y^2 = 1 - z^2$
$x^2 - y^2 = z^2 + 1$

f) Parametrize the surface $x^2 + y^2 - z^2 = 1$ as

$$r(\theta, z) = [\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, \ldots\ldots]^T \ .$$

g) Who was the creative person who discovered dark matter and proposed the mechanism of gravitational lensing?

h) What is the cosine of the angle between the matrices $A, B \in M(2, 2)$, where A is the identity matrix and B is the matrix which has 1 everywhere? You should get a concrete number.

i) We have seen the identity $|v|^2 + |w|^2 = |v - w|^2$, where v, w are vectors in $\mathbb{R}^n$. What conditions do v and w have to satisfy so that the identity holds?

j) Compute the exterior derivative dF of the differential form

$$F = e^x \sin(y) dxdy + \cos(xyz) dydz \ .$$

Solution:
a) $f_{xx} < 0$.
b) $m = 2, n = 3$.
c) $\rho^3 \sin(2\phi)/2$.
d) All except $[0, x]$.
e) Only one: $x^2 - y^2 = 1 - z^2$.
f) $r(\theta, z) = [\sqrt{z^2+1}\cos(\theta), \sqrt{z^2+1}\sin(\theta)z]$.
g) Fritz Zwicky.
h) $tr(A^T B)/(\sqrt{2}2) = 1/\sqrt{2}$
i) perpendicular
$-yz\sin(xyz)dxdydz$.

Problem 39B.4) (10 points):
a) (4 points) Find the plane Σ which contains the three points

$$A = (3, 2, 1), \quad B = (3, 3, 2), \quad C = (4, 3, 1) \ .$$

b) (3 points) What is the area of the triangle ABC?

c) (3 points) Find the distance of the origin $O = (0, 0, 0)$ to the plane Σ.

Solution:
a) The normal vector to the plane is $n = AB \times AC$ which is $n = [-1, 1, -1]$. The equation of the plane is $-x + y - z = -2$.
b) The area is $\sqrt{3}/2$.
c) The distance is $|AO \cdot n|/\sqrt{n} = |[3, 2, 1] \cdot [-1, 1, -1]|/\sqrt{3} = 2/\sqrt{3}$.

Problem 39B.5) (10 points):
a) (8 points) Find all the critical points of the function

$$f(x, y) = x^5 - 5x + y^3 - 3y$$

and classify these points using the second derivative test.

b) (2 points) Is any of these points a global maximum or global minimum of f?

Solution:
a) The critical points solve $5x^4 - 5 = 0$ and $3y^2 - 3 = 0$. There are 4 critical points $(1, 1), (1, -1), (-1, 1), (-1, -1)$. The discriminant is $120x^3y$. It is negative for $(1, -1), (-1, 1)$ so that these are saddle points. At $(1, 1)$, we have $f_{xx} = 20x^3 = 1$ which shows that $(1, 1)$ is a local minimum. The other point $(-1, -1)$ is a local maximum.
b) There is no global maximum, nor global minimum.

Problem 39B.6) (10 points):
a) (8 points) Use the Lagrange method to find **all the maxima and all the minima** of

$$f(x,y) = x^2 + y^2$$

under the constraint

$$g(x,y) = x^4 + y^4 = 16 \ .$$

b) (2 points) In our formulation of Lagrange theorem, we also mentioned the case, where $\nabla g(x,y) = [0,0]^T$. Why does this case not lead to a critical point here?

Solution:
a) The Lagrange equations are

$$\begin{aligned} c2x &= \lambda(4x^3) \\ 2y &= \lambda(4y^3) \\ x^4 + y^4 &= 16 \end{aligned}$$

Distinguish the cases $x = 0, y = 0$. This leads to the 4 critical points on the axes $(\pm 2, 0), (0, \pm 2)$. Then, if x, y are both not zero, then there are more critical points They are located on $(\pm a, \pm a)$, where $a = 8^{1/4}$.
To see which are the maxima and minima, just look at the function values. For $x = 2, y = 0$ we get the value 4. For $x = y = 8^{1/4}$ we get $x^2 + y^2 = 2\sqrt{8}$ which is larger.
b) We are not on the constraint. So this condition is not relevant.

Problem 39B.7) (10 points):
a) (5 points) The hyper surface

$$S = \{f(x,y,z,w) = x^2 + y^2 + z^2 - w = 5\}$$

defines a three-dimensional manifold in $\mathbb{R}^4$. It is poetically called a **hyper-paraboloid**. Find the tangent plane to S at the point $(1,2,1,1)$.

b) (5 points) What is the linear approximation $L(x,y,z,w)$ of $f(x,y,z,w)$ at this point $(1,2,1,1)$?

Solution:
a) The gradient is $[2x, 2y, 2z, -1]$ The hyperplane has the equation $2x + 4y + 2z - w = 11$.
b) The linearization is $5 + 2(x-1) + 4(y-2) + 2(z-1) - 1(w-1)$.

Problem 39B.8) (10 points):
Estimate the value $f(0.1, -0.02)$ for

$$f(x, y) = 3 + x^2 + y + \cos(x + y) + \sin(xy)$$

using quadratic approximation.

Solution:
There are two ways. We can just compute $f(0, 0) = 4$, $f_x(0, 0) = 0$, $f_y(0, 0) = 1$, $f_{xx}(0, 0) = 1/2$, $f_{yy}(0, 0) = -1/2$, $f_{xy}(0, 0) = 0$ and then write down $Q(x, y) = 4 + y + x^2/2 - y^2/2$ then plug in the points $x = 0.1, y = -0.02$ which gives 3.9848. This is very close. The error is about $1.7 \cdot 10^{-6}$. The other way is to recall the Taylor series of $\cos(u)$ and $\sin(u)$, to get $3 + x^2 + y + 1 - (x + y)^2/2 + xy$ after neglecting terms of higher order. This gives the same thing.

Problem 39B.9) (10 points):
a) (8 points) We vacation in the **5-star hotel** called **MOTEL 22** in 5-dimensional space and play there ping-pong. The ball is accelerated by gravity $r''(t) = [x(t), y(t), z(w), v(t), w(t)] = [0, 0, 0, 0, -10]^T$. We hit the ball at $r(0) = [4, 3, 2, 1, 2]^T$ and give it an initial velocity $r'(0) = [5, 6, 0, 0, 3]^T$. Find the trajectory $r(t)$.

b) (2 points) At which positive time $t > 0$ does the ping-pong ball hit the **hyper ping-pong table** $w = 0$? (The points in this space are labeled $[x, y, z, v, w]$.)

Solution:
a) Just integrate twice. We get

$$r(t) = [5t + 4, 6t + 3, 2, 1, -5t^2 + 3t + 2] \ .$$

b) When is $-5t^2 + 3t + 2 = 0$? This is a quadratic equation and we see that $t = 1$ is the only solution in positive time.

Problem 39B.10) (10 points):
a) (5 points) Integrate the function $f(x, y) = (x^2 + y^2)^{22}$ over the region $G = \{1 < x^2 + y^2 < 4, y > 0\}$.

b) (5 points) Find the area of the region enclosed by the curve

$$r(t) = [\cos(t), \sin(t) + \cos(2t)]^T \ ,$$

with $0 \leq t \leq 2\pi$.

Solution:
a) Integrate in polar coordinates:

$$\int_1^2 \int_0^\pi r^{44} r d\theta dr = \pi(2^{46}-1)/46 .$$

b) Use Greens theorem with $F = [0, x]$. This gives the line integral

$$\int_0^{2\pi} \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) - 2\sin(2t) \end{bmatrix} dt = \pi .$$

Problem 39B.11) (10 points):
a) (7 points) Integrate

$$f(x, y, z) = x^2 + y^2 + z^2$$

over the solid

$$G = \{x^2 + y^2 + z^2 \leq 4, z^2 < 1\} .$$

b) (3 points) What is the volume of the same solid G?

Solution:
This appeared to be toughest problem for the class as only a handful of students could solve it. It was maybe psychological, as one was tempted to use spherical coordinates. What is always good is to make a picture. Without a picture, it appears as if the region can be well expressed in spherical coordinates. It can but one has to split up the integral. The problem is quite easily solved in polar (cylindrical) coordinates:
a)

$$\int_0^{2\pi} \int_{-1}^1 \int_0^{\sqrt{4-z^2}} (z^2 + r^2) r dr dz d\theta$$

This is

$$2\pi \int_{-1}^1 \frac{1}{2} z^2 \left(4 - z^2\right) - \frac{1}{4} z^2 \left(4 - z^2\right)^2 \, dz = 43\pi/7 .$$

b)

$$\int_0^{2\pi} \int_{-1}^1 \int_0^{\sqrt{4-z^2}} r dr dz d\theta$$

Which is $2\pi \int_{-1}^1 (4 - z^2)/2 = \pi 22/3$.

Problem 39B.12) (10 points):
a) (8 points) Compute the line integral of the vector field

$$F = [yzw + x^6, xzw + y^9, xyw - z^3, xyz + w^4]^T$$

along the path

$$r(t) = [t + \sin(t), \cos(2t), \sin(4t), \cos(7t)]^T$$

from $t = 0$ to $t = 2\pi$.

b) (2 points) What is $\int_0^{2\pi} r'(t)\, dt$?

Solution:
a) This is a gradient field with potential $f(x, y, z) = xyzw + x^7/7 + y^{10}/10 - z^4/4 + w^5/5$. The **fundamental theorem of line integrals** gives the result $f(r(2\pi)) - f(r(0))$ which is $(2\pi)^7/7$.
b) This is just integrating a vector valued function. It is $r(2\pi) - r(0)$ by the fundamental theorem of calculus. The result is $[2\pi, 0, 0, 0]$.

Problem 39B.13) (10 points):
a) (8 points) Find the line integral of the vector field

$$F(x, y) = [3x - y, 7y + \sin(y^4)]^T$$

along the polygon $ABCDE$ with $A = (0, 0), B = (2, 0), C = (2, 4), D = (2, 6), E = (0, 4)$. The path is closed. It starts at A, then reaches B, C, D, E until returning to A again.

b) (2 points) What is line integral if the curve is traced in the opposite direction?

Solution:
a) This is a problem for Greens theorem. The curl is 1 so that the line integral is just the area of the region, which is 10.
b) Reversing the order of the path changes the sign of the result which is -10.

Problem 39B.14) (10 points):
a) (8 points) What is the flux of the vector field

$$F(x, y, z) = [y + x^3, z + y^3, x + z^3]^T$$

through the sphere $S = \{x^2 + y^2 + z^2 = 9\}$ oriented outwards?

b) (2 points) What is the flux of the same vector field F through the same sphere S but where S is oriented inwards?

Solution:
a) This is a problem for the divergence theorem. The divergence is $3\rho^2$.

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{3} 3\rho^2 \rho^2 \sin(\phi) \; d\rho d\phi d\theta \; .$$

This is $2\pi \cdot \cdot 2 \cdot 3 \cdot 3^5/5 = 12\pi 3^5//5$
b) Changing the order of integration gives a negative result.

Problem 39B.15) (10 points):
a) (7 points) What is the flux of the curl of the vector field

$$F(x, y, z) = [-y, x + z(x^2 + y^5), z]^T$$

through the surface

$$S = \{x^2 + y^2 + z^2 + z(x^4 + y^4 + 2\sin(x - y^2 z)) = 1, z > 0\,\}$$

oriented upwards?

b) (3 points) The surface in a) was not closed, it did not include the bottom part

$$D = \{z = 0, x^2 + y^2 \leq 1\} \; .$$

Assume now that we close the bottom and orient the bottom disc D downwards. What is the flux of the curl of the same vector field F through this closed surface obtained by taking the union of S and D?

Solution:
a) This is a situation for Stokes theorem. We parametrize the boundary as $r(t) = [\cos(t), \sin(t), 0]$ then compute the line integral. It is 2π.
b) The result is now zero, because we integrate the curl of a vector field over a closed surface.