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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Welcome to the final exam. Please don't get started yet. We start all together at 9:00 AM after getting reminded about some formalities. You can fill out the attendance slip already. Also, you can already enter your name into the larger box above.

- You only need this booklet and something to write. Please stow away any other material and any electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 2 and 3 we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is additional space on the back of each page. If you must, use additional scratch paper at the end. But put your final result near the question and box the final result.
- If you finish a problem somewhere else, please indicate on the problem page where we can find it.
- You have 180 minutes for this final exam.

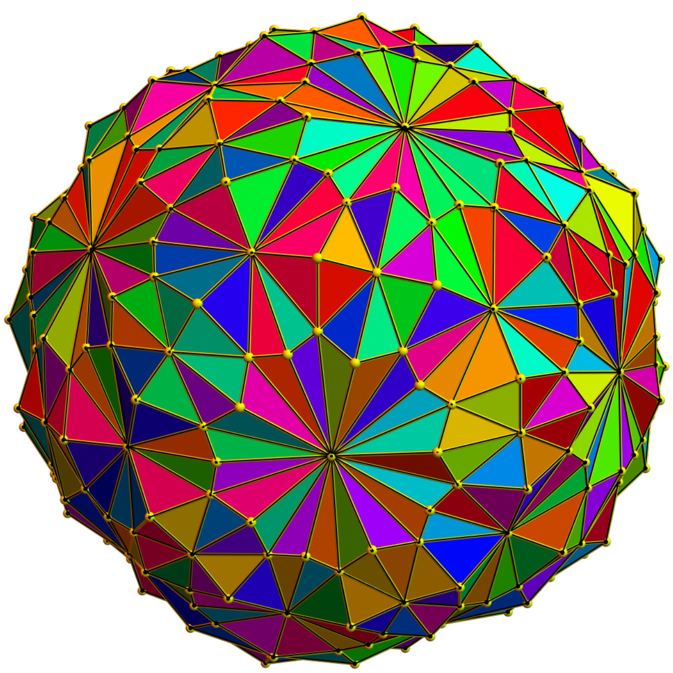


FIGURE 1. A two dimensional discrete sphere S.

Unit 39: Final Exam

Problems

Problem 39.1) (10 points):
In Figure 2 (see the next page for a larger version) you see a discrete two dimensional region G in which all triangles are oriented counter clockwise. The one-form F as a function on oriented edges is given in the picture. Answer the following questions and give reasons:

a) (2 points) The curl dF of F is a function on oriented triangles. What can you say about the sum over all the curl values dF in the graph G of Figure 2?

b) (2 points) Is F a gradient field $F = df$ for some function f on vertices?

c) (2 points) What is the sum of the natural divergence values d^*F on vertices?

d) (2 points) What was the name of the matrix $K = d^*d$ that acts on 0-forms. It has been defined more than 150 years ago.

e) (2 points) In Figure 1 on the front page, you saw a two-dimensional discrete sphere S. which plays the role of a closed surface $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Given a 1-form F, a function on oriented edges of S, what is the sum over all curls on S? The answer is a number but you have to justify the answer.

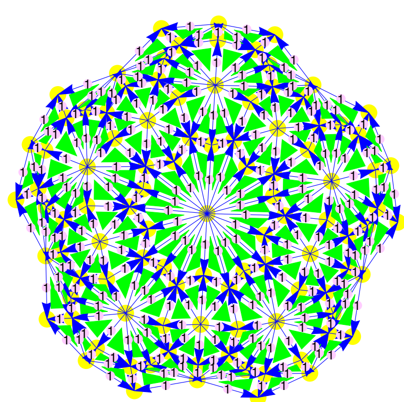
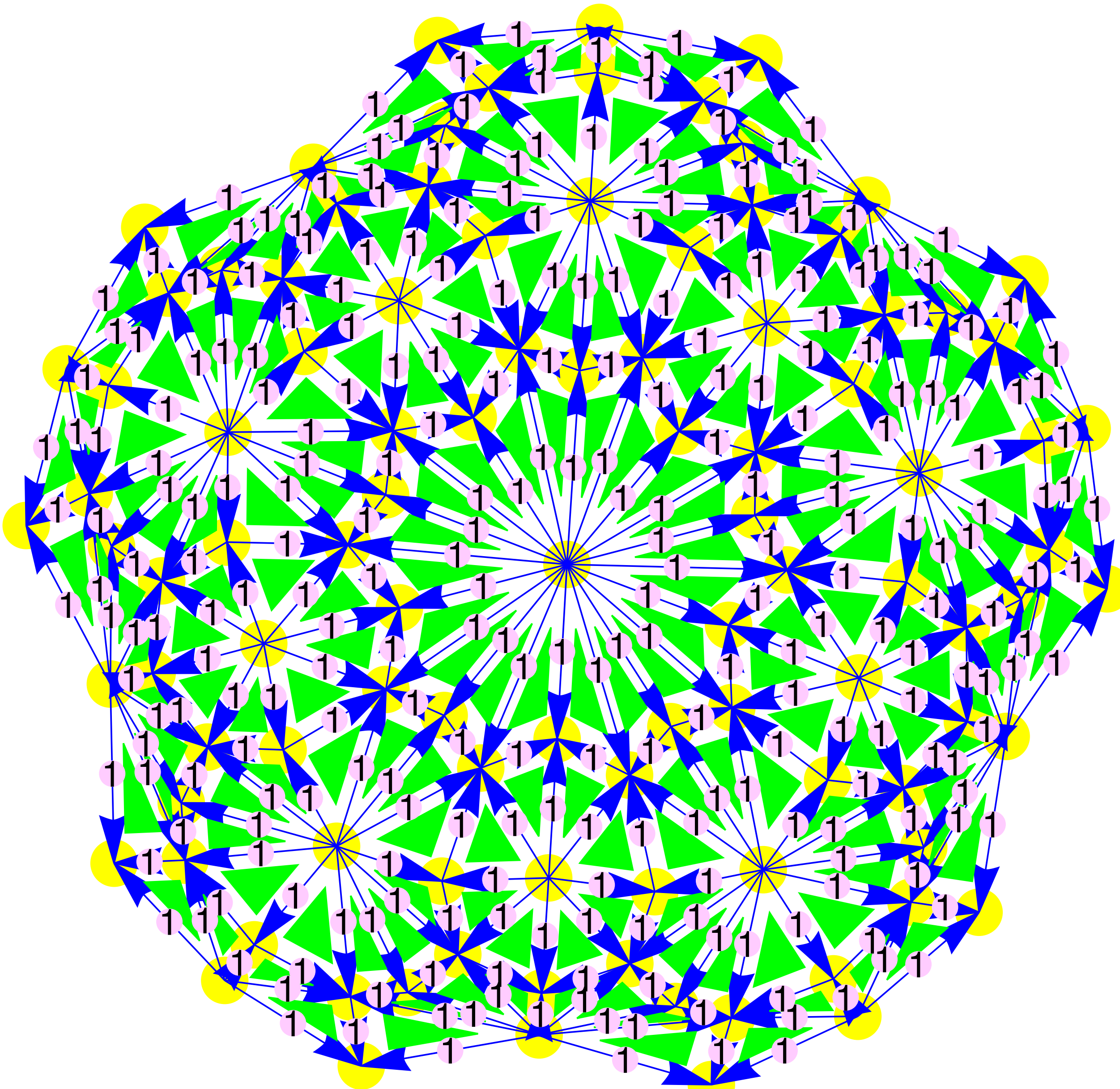


Figure 2.



An enlargement of Figure 2, showing the region G from problem 39.1.

Solution:
Solution: a) By the discrete Stokes rsp Green theorem, the sum of all curls is equal to the line integral along the boundary, which is zero.
b) This was often done wrong. The fact that one line integral is zero does not mean it is a gradient field. All line integrals have to be zero, especially line integrals along triangles which is called the curl. The curl is actually nowhere zero.
c) By the natural divergence theorem (Kirchhoff's law) the sum over all divergences is zero.
d) It is called the Kirchhoff matrix.
e) The sum of the curls is zero because the surface is closed. The surface does not have a boundary. By Stokes theorem, the sum of all curl values is equal to the line integral over the empty set which is zero.

Problem 39.2) (10 points) Each question is one point:

a) Albert Einstein used the notation $v_k w^k$ for two vectors v, w. It is today called "Einstein notation". What did Einstein mean, when he wrote $v_k w^k$?

b) If $S = r(R)$ is a two-dimensional surface parametrized by $r(u, v) = [x(u, v), y(u, v), z(u, v)]^T$, what is the relation between $|r_u \times r_v|$ and $\sqrt{\det(dr^T dr)}$?.

c) What is the Newton method used for? We have seen this numerical tool in a proof seminar.

d) What is the curvature of a circle with radius 20?

e) Define the 1×5 matrix $A = [1, 1, 1, 1, 1]$. One of the two matrices $A, B = A^T$ is row reduced. Which one?

f) What is the distortion factor of the coordinate change $\Phi(x, y) = (3x + y, x + y)$?

g) What is the numerical value of i^{22}, if $i = \sqrt{-1}$ is the imaginary unit?

h) What is the name of the differential equation $i\hbar \frac{d}{dt}\psi = K\psi$, where K is a matrix? It appears in a theory which also is called "matrix mechanics".

i) Why is the distance between two lines $r_1(t) = Q + tv$ and $r_2(t) = P + tw$ given by the formula $|(v \times w) \cdot PQ|/|v \times w|$?

j) You are given a Morse function f on a 2-torus and you count that f has 11 maxima and 11 minima. How many saddle points are there?

Herr Einstein wishes you good luck!

Solution:
a) Dot product
b) Equal
c) Finding roots
d) 1/20
e) A
f) the determinant 3-1=2
g) -1
h) Schroedinger equation
i) Volume/area
j) 22 as the sum is zero.

Problem 39.3) (10 points) Each question is one point:
In this problem, we work in hyperspace $\mathbb{R}^4$, where points have coordinates (x, y, z, w).

a) Write down the exterior derivative dF of the 2-form

$$F = x^2y^2z^2w^2dydz \; .$$

b) Write down the exterior derivative of the 3-form

$$F = x^2y^2z^2w^2dxdzdw \; .$$

c) Let G be the two-dimensional torus $x^2 + y^2 = 1, z^2 + w^2 = 1$ embedded in $\mathbb{R}^4$. What does the general Stokes theorem tell about $\iint_G F \; dS$, where F is the 2-form from a)?

d) What is $d^2F = ddF$, where F is the 2-form given in a)?

e) What is $d^2F = ddF$, where F is the 3-form given in b)?

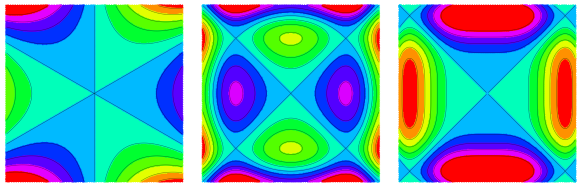
f) A $(1, 1)$ tensor on $\mathbb{R}^4$ can be interpreted as a 4×4 ....

g) A $(0, 1)$ tensor on $\mathbb{R}^4$ can also be interpreted as a ....

h) Is grad(grad(f)) defined if f is a function?

i) Does div(div(F)) make sense for any field F?

j) You see 3 contour maps of functions f, g and h of two variables. One of them is not Morse. Which one? The first the second or the third?



Solution:
a) $dF = 2xy^2z^2w^2 dxdydz + 2wx^2y^2z^2 dwdydz$.
b) $dF = 2yx^2z^2w^2 dydxdzdw$.
c) This is a triple integral $\int\int_G dF$ of dF over a solid G has the two-dimensional torus as boundary. This is not necessarily zero.
d) 0 (done in homework)
e) 0 (done in homework)
f) This is equivalent to a matrix A. Indeed, a matrix defines a bi-linear map $(v, w) \to vAw$ taking a row vector v and a column vector w and getting a real number (and so by definition is a $(1, 1)$ tensor).
g) This is a row vector v. A row vector defines a linear map from the space of column vectors to the reals. The map is the matrix product $w \to vw$.
h) No, grad is an operation which gives from a 0-form a 1-form. You can not apply grad on a 1-form.
i) No, div is an operation which gives from a $(n-1)$-form in n dimensions a n-form in n dimensions. You can not apply divergence again. Similarly we have seen the natural divergence which is the adjoining of the gradient and which defines from a 1-form a 0-form. But also this can not be applied twice. We can apply divgrad for example if we identify 1-forms and 2-forms. We have also seen it making sense with the "natural divergence" $grad^*$, as then it is $d^*d = grad^*grad = divgrad$. This is how calculus books do it and how it is also implemented in the discrete when getting the Kirchhoff matrix.
j) It is the first one. If three level surfaces intersect we have a critical point which is not of the 3 possible Morse types maximum, minimum and saddle point. By the way, one can look at level surfaces and so contour maps also in the discrete. (See https://arxiv.org/abs/1508.05657). I was not joking when predicting that in 2072 when your grand kids are in college, calculus will no more be taught using old-fashioned "limits". Limits are so lame!

Problem 39.4) (10 points):
a) (3 points) Parametrize the line L which contains the points

$$A = (3, 2, 1), \quad B = (3, 3, 2) .$$

b) (3 points) Given the additional point $P = (3, 3, 3)$, find the distance between P and L.

c) (4 points) Write down the equation $ax + by + cz = d$ of the plane containing L and P.

Solution:
a) $r(t) = [3, 2, 1] + t(B - A) = [3, 2 + t, 1 + t]^T$
b) We use the area/base formula to get $|[0, 1, 1] \times [0, 0, 1]|/|[0, 1, 1]| = 1/\sqrt{2}$.
c) The cross product of $AB \times AP = [1, 0, 0]^T = [a, b, c]$ is perpendicular to the plane. This gives $x = d$. Plug in the point like A or B to $x = 3$.
Remark. The problem was constructed with the intention to have the computations reasonable. This of course risks allowing short cuts. While we like **clever short solutions**, we wanted to test here multi-variable concepts and not high school geometry. One can notice for example that all x-coordinates of the point are 3, we can write down the plane x=3 immediately. Then we can use a two dimensional situation and draw out the points $(2, 1), (3, 2)$ and $(3, 3)$ figuring out the distance like a 9th grader. We point this out in an other problem again: you do not gain much sympathy with “smarty-pant” solutions without demonstrating that you master the general technique. The best way to handle this as a student to solve it the standard way (demonstrating that you can do the general case and for example could write down a computer program which does the general case) and then say: “By the way, the problem can also be solved with school geometry by noticing ...”. This assures that everybody applauds, also the grader. Here, the problem was designed to have low complexity. A case $A = (13, 32, 31), \quad B = (7, 9, 11), P = (13, 17, 2)$ could not have been hi-jacked but would produce nasty integer arithmetic.

Problem 39.5) (10 points):
a) (6 points) Find all the critical points of the function

$$f(x, y) = x^7 - 7x + xy - y$$

and classify them using the second derivative test.

b) (2 points) The island theorem told us that the number of maxima plus the number of minima minus the number of saddle points of f is 1 on an island. In the current case this fails. Why does this not contradict the island theorem?

c) (2 points) Does the function f have a global maximum or a global minimum?

Solution:
a) This is a standard problem. First write down the gradient $\nabla f(x, y) = [7x^6 - 7 + y, x - 1]^T = [0, 0]^T$ leading to $(x, y) = (1, 0)$. Then compute the Hessian matrix and compute $f_{xx}f_{yy} - f_{xy}^2 = -1$ showing that the critical point is a saddle.
b) The entire plane $\mathbb{R}^2$ is not a bounded island.
c) Take x=0 for example, to get the function $f(x, y) = -y$ which does not have an upper or lower bound.

Problem 39.6) (10 points):
a) (7 points) Use the Lagrange method to find the minimum of the function

$$f(x, y, z, w) = x^2 + 2y^2 + 3z^2 + w^2$$

under the constraint

$$g(x, y, z, w) = x + y + z + w = 17.$$

b) (3 points) You saw in a) that in this case, the Lagrange equations are a system of linear equations for a couple of unknown. This can be written in matrix form as $AX = b$, where the vector X encodes the unknown quantities and b is a constant vector. What is the size of the matrix A?

Solution:
a) Write down the Lagrange equations. These are 5 equations which turn out in this case to be linear and are easy to solve after eliminating λ. The minimum is at $(6, 3, 2, 6)$.
b) The Lagrange equations for (x, y, z, w, λ) is here given as a linear system $Ax = b$ for a 5×5 matrix. Note that the Lagrange multiplier λ is also an unknown.

Problem 39.7) (10 points):
a) (5 points) Find the tangent plane at the point $P = (3, 1, 3, -1)$ of the **hyper cone**

$$S = \{f(x, y, z, w) = x^2 + y^2 - z^2 - w^2 = 0\}$$

in $\mathbb{R}^4$.

b) (5 points) Write down the linearization $L(x, y, z, w)$ of $f(x, y, z, w)$ at $(3, 1, 3, -1)$.

Solution:
a) $6x + 2y - 6z + 2w = 0$.
b) $L(x, y, z, w) = 6(x - 3) + 2(y - 1) - 6(z - 3) + 2(w + 1)$

Problem 39.8) (10 points):
Estimate the value $f(0.1, -0.02)$ for $f(x, y) = e^{x+y}$ using quadratic approximation $Q(x, y)$ at $(x_0, y_0) = (0, 0)$.

Solution:
Compute the gradient $\nabla f(x, y) = \begin{bmatrix} e^{x+y} \\ e^{x+y} \end{bmatrix}$ and the Hessian $H(x, y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{bmatrix}$ which at the point $(0, 0)$ are $\nabla f(0, 0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $H(0, 0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This gives $L(x, y) = 1 + x + y$ and $Q(x, y) = 1 + x + y + x^2/2 + y^2/2 + xy$. Plugging in the values into the quadratic approximation ! gives 1.0832.
Remark: There is a short cut which a few students have used (I have mixed feelings about this as we try to make the problems so simple that they can be solved in reasonable time the risk is that the problem can be **hi-jacked with single variable**. In general, we want to test what you have learned in the current course. A problem like $f(x, y) = \sqrt{y * \sin(xe^{x+xy})}$ not allowing a short cut would have been painful. Most sympathies are gained from the grader by solving it the way intended and then say: **"By the way, the problem can also be solved by noting $e^u = 1 + u + u^2/2! + ...$ and plugging in $u = x + y$."** Or, alternatively **"By the way, the problem can also be solved by writing $e^x = 1 + x + x^2/2! + ...$ and $e^y = 1 + y + y^2/2! + ...$, then writing $e^{x+y} = (1 + x + x^2/2! + ...)(1 + y + y^2/2! + ...)$ and neglecting terms of order 3 and higher"**. Still, also show that you know what a "gradient" and what a "Hessian" and demonstrate you understand what the Taylor theorem in higher dimensions.

Problem 39.9) (10 points):

a) (6 points) Find the curve $r(t)$ which satisfies $r(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $r'(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $r''(t) = \begin{bmatrix} 1-\sin(t) \\ -4\sin(2t) \\ -9\sin(3t) \end{bmatrix}$.

b) (4 points) What is the curvature of the curve at the point $r(0)$?

Solution:
a) $r(t) = [t^2/2 + \sin(t), \sin(2t), \sin(3t)]^T$.
b) In the formula $|r'(0) \times r''(0)|/|r'(0)|^3$ both vectors are known. This is $\sqrt{13/14^3}$.

Problem 39.10) (10 points):
Find the area of the region enclosed by the curve

$$r(t) = \begin{bmatrix} 3\cos(t) \\ 2\sin(t) + \cos(7t) \end{bmatrix},$$

where $0 \le t \le 2\pi$.

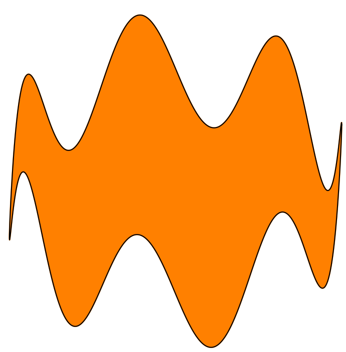


FIGURE 3. The region in problem 39.10.

Solution:
Use Green's theorem with $[0, x]^T$ as vector field to get 6π. For the integral, use the symmetry that if an odd 2π- periodic function is integrated over $[0, 2\pi]$ this gives zero. And a double angle formula is needed to integrate $\cos^2(x)$. But we have this done a thousand times.

Problem 39.11) (10 points):
Integrate

$$f(x,y,z) = \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}$$

over the half avocado

$$E = \{4 \leq x^2+y^2+z^2 \leq 16, z \leq 0\} \ .$$

In other words, compute $\iiint_E f \ dV$.

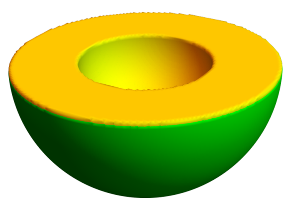


FIGURE 4. The avocado in problem 39.11.

Solution:
We use spherical coordinates of course.

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_2^4 \frac{e^{\rho^2}}{\rho} \rho^2 \sin(\phi) \ d\rho d\phi d\theta \ .$$

The result is $\pi(e^{16} - e^4)$.

Problem 39.12) (10 points):
Compute the line integral

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt$$

of the vector field

$$F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 3x^2 + yz \\ 3y^2 + xz \\ 3z^2 + xy \end{bmatrix}$$

along the path C parametrized by

$$r(t) = \begin{bmatrix} \cos(7\pi t)e^{t(1-t)} \\ \sin(11\pi t) \\ e^{t(1-t)} \end{bmatrix}$$

from $t = 0$ to $t = 1$.

Solution:
Use the FTLI. The potential is $f(x, y, z) = x^3 + y^3 + z^3 + xyz$. Now compute $A = r(0)$ and $B = r(1)$ to get $f(B) - f(A) = -2$.

Problem 39.13) (10 points):
Find the line integral $\int_C F \cdot dr$ of the vector field

$$F(x, y) = \begin{bmatrix} y + x^4 \\ y^3 + y^4 \end{bmatrix}$$

along the boundary C of the hexagon region shown in the picture. The curve C is a closed polygon going counter clockwise from $(2, 0)$ over $(1, 2), (-1, 2), (-2, 0), (-1, -2), (1, -2)$ back to $(2, 0)$.

Solution:
Use Green's theorem again. The curl(F) is constant -1. The area is 12. The result is -12. Quite many missed that problem. Possibly because it never happend before in any of my exams that two times the same integral theorem was used. Knill, the sneaky bastard!

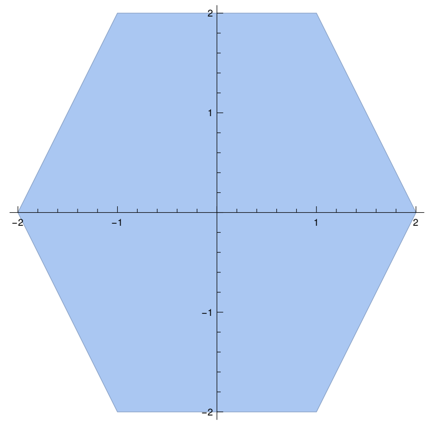


FIGURE 5. The hexagon in Problem 39.13.

Problem 39.14) (10 points):
Find the flux $\iint_S \text{curl}(F) \cdot dS$ of the curl of the vector field

$$F = \begin{bmatrix} x^7 \\ -x \\ \sin(z^2) + z^3 x \end{bmatrix}$$

through the surface S parametrized by

$$r(s,t) = \begin{bmatrix} (6 + 2\cos^2(s/2)\cos(t))\cos(2s) \\ 2\cos^2(s/2)\sin(t) + 2s \\ (6 + 2\cos^2(s/2)\cos(t))\sin(2s) \end{bmatrix}$$

with $0 \leq s \leq 7\pi/2$ and $0 \leq t < 2\pi$. **Hint:** The surface has two boundary curves obtained by looking at $s = 0$ or $s = 7\pi/2$. We don't tell you the orientation of the larger curve

$$r_1(t) = r(0,t) = [6 + 2\cos(t), 2\sin(t), 0]^T$$

is but you should know that the smaller curve

$$r_2(t) = r(7\pi/2, t) = [-6 - \cos(t), \sin(t) + 7\pi, 0]^T$$

is correctly oriented.

Solution:
Of course, this is a Stokes theorem. It had been written all over the wall in this problem. The line integral along the smaller one is π. The line integral along the larger one is -4π. but the later is oriented in the wrong direction. The result is 5π.

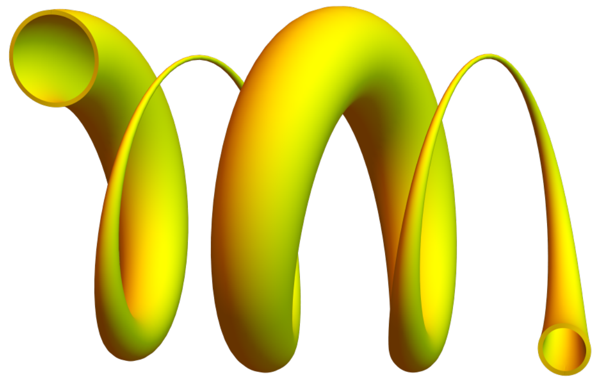


FIGURE 6. The surface S with two boundary circles in Problem 39.14.

Problem 39.15) (10 points):
Find the flux

$$\iint_S F \cdot dS$$

of the vector field

$$F = \begin{bmatrix} \sin(z) + y^3 + x \\ \sin(x) + z^3 + y \\ \sin(y) + x^3 + z \end{bmatrix}$$

through the boundary surface S of the solid E given in the picture. The solid is obtained by sculpuring a cube $-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1$ of side length 2, by cutting away at each corner the points in distance less than 1 from that corner. In other words, we look at the points in the cube which have distance larger than 1 from any of the 8 corners. The surface S bounding the solid E is oriented outwards.

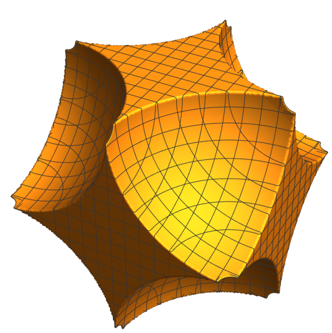


FIGURE 7. The solid given in Problem 39.15.

Solution:
Use the divergence theorem. The divergence is constant 3. The result is 3 times the volume of the solid, which is $3(8 - 4\pi/3)$. Note that each corner is $1/8$ of a sphere and there are 8 corners so that we have to subtract $4\pi/3$ from 8.

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