ECON 2010C — Problem Set 1 – Suggested Solutions

Borui N. Zhu

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This is the first time this set of problems is circulated at this department, so all solutions are new. If you catch an error, please let me know. Thanks! I do prefer the proper way of spelling though ("summarise" instead of "summarize" etc).

1 Question 1: Calibrating the Growth Model

Consider the following economy, the equilibrium of which solves the following social planner's problem:

$$max \sum_{t=0}^{\infty} \beta^t N_t[\log(c_t) - Bh_t],$$

subject to

$$N_t(c_t + i_t) = (N_t k_t)^{\theta}(\gamma^t N_t h_t)^{1-\theta}, \quad \gamma \geq 1, \ 0 < \theta < 1$$
$$N_{t+1}k_{t+1} = (1-\delta)N_t k_t + N_t i_t, \quad 0 < \delta < 1$$

where $h_t \geq 0$, population growth is given by $N_{t+1} = \eta N_t$, for $\eta \geq 1$, and k_0 and N_0 are given. Here c_t, i_t, h_t and k_t are consumption per capita, investment per capita, hours worked per capita and capital stock per capita.

(a) **Write the Bellman equation for this problem.**

The trouble here is γ^t, which causes an explicit time-dependence on t. The whole point of writing things recursively is to get rid of the time dimension. We observe that $A_{t+1} \equiv \gamma^{t+1} = \gamma\gamma^t = \gamma A_t$, which is a time-independent relationship. So one trick is to introduce another state variable with the appropriate law of motion:

$$\begin{aligned} V(A, N, k) = \max_{c,i,h,k'} &N\big[log(c) - B \cdot h\big] + \beta V(N', A', k') \quad s.t. \\ &N(c+i) = (Nk)^{\theta}(ANh)^{1-\theta} \\ &N'k' = (1-\delta)k + Ni \\ &A' = \gamma A \\ &N' = \eta N \end{aligned}$$

We can use the tools from David's class to solve it.

But now let's solve part (b) as a sequential problem this time. The sequential formulation of the problem is:

$$\max_{\{c_t,h_t,i_t,k'_{t+1}\}} \sum_{t=0}^{\infty} \beta^t N_t\big[log(c_t) - Bh_t\big] \quad s.t.\ \forall t:$$
$$N_t(c_t + i_t) = (N_t k_t)^\theta(\gamma^t N_t h_t)^{1-\theta}, \quad \gamma \geq 1,\ \ 0 < \theta < 1$$
$$N_{t+1}k_{t+1} = (1-\delta)N_t k_t + N_t i_t, \quad 0 < \delta < 1$$
$$k_0 > 0 \text{ given}$$
$$N_{t+1} = \eta N_t \quad \eta \geq 1$$

(b) **Solve for the balanced growth path of this economy.**

Set up the Lagrangian problem, where we solve one of the constraints for $N_t i_t$ (getting rid of i_t as choice variable) and plug it into the other:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t N_t\big[\log(c_t) - Bh_t\big] + \lambda_t\Big[(N_t k_t)^\theta(\gamma^t N_t h_t)^{1-\theta} - N_t c_t - N_{t+1}k_{t+1} + (1-\delta)N_t k_t\Big]$$

Solve for FOCs to obtain:

$$[c_t] : \beta^t N_t \frac{1}{c_t} = \lambda_t N_t$$
$$[h_t] : \beta^t N_t B = \lambda_t (N_t k_t)^\theta (\gamma^t N_t)^{1-\theta}(1-\theta)h_t^{-\theta}$$
$$[k_{t+1}] : \lambda_t N_{t+1} = \lambda_{t+1}\Big[N_{t+1}^\theta \theta k_{t+1}^{\theta-1}(\gamma^{t+1}N_{t+1}h_{t+1})^{1-\theta} + (1-\delta)N_{t+1}\Big]$$

Cancel out the level of population N_t wherever possible, use $[c_t]$ to get rid of the multiplier $\lambda_t = \beta^t \frac{1}{c_t}$, and use $N_{t+1} = \eta N_t$:

$$[h_t] : \beta^t B = \beta^t \frac{1}{c_t} k_t^\theta (\gamma^t)^{1-\theta}(1-\theta)h_t^{-\theta}$$
$$[k_{t+1}] : \beta^t \frac{1}{c_t} = \beta^{t+1}\frac{1}{c_{t+1}}\Big[\theta k_{t+1}^{\theta-1}(\gamma^t h_{t+1})^{1-\theta} + (1-\delta)\Big]$$

For optimality, we see that population was more annoying than critical in any way. Simplify one more time:

$$[h_t] : B = \frac{1}{c_t} k_t^\theta (\gamma^t)^{1-\theta}(1-\theta)h_t^{-\theta} \qquad \text{(consumption-leisure trade-off)}$$
$$[k_{t+1}] : \frac{1}{c_t} = \beta\frac{1}{c_{t+1}}\Big[\theta k_{t+1}^{\theta-1}(\gamma^{t+1}h_{t+1})^{1-\theta} + (1-\delta)\Big] \qquad \text{(Euler equation)}$$

These conditions capture the trade-offs the planner faces: The consumption labour choice weights the disutility of labour B on the left-hand side against marginal productivity of labour today on the right-hand side. The Euler equation is the optimality condition for intertemporal choice, weighting consumption today – via marginal utility – on the left-hand side against consumption tomorrow on the right-hand side. This intertemporal substitution works via capital as savings device, where we need to consider its depreciation and future marginal productivity: that forms the square bracket on the right-hand side.

A balanced growth path is defined as a sequence of quantities that grow at an equal growth rate. So $c_t = g^c c_{t-1}$, $h_t = g^h h_{t-1}$, $i_t = g^i i_{t-1}$, $k_t = g^k k_{t-1}$. We conjecture that $g^c = g^i = g^k = g$ and $g^h = 0$. The reason for this guess is as follows: We see that γ makes labour more productive every period – it increases "effective labour". Capital accumulates endogenously, so we expect it to grow at the same rate (otherwise we would have a more and more skewed capital/effective labour-ratio, which does not seem optimal). Capital accumulation mechanically translates into investment rates, so we expect the same rate for i. h is more tricky: Households dislike working, so it would be unintuitive to have the hours of work growing, especially since γ makes effective labour input grow given constant hours. A guess is that either $0 < g^h < 1$ or $g^h = 0$. Let's go with the latter and see what happens.

If we plug in this assumption and get a contradiction, that means either that our guess was wrong or that there is no balanced growth path (BGP).

Plugging our conjecture into optimaliy conditions and relevant constraints:

$$[h_t] : B = \frac{1}{g^t c_0}(g^t k_0)^\theta (\gamma^t)^{1-\theta}(1-\theta)h_0^{-\theta} \qquad \text{(consumption-leisure trade-off)}$$

$$[k_{t+1}] : \frac{1}{g^t c_0} = \beta \frac{1}{g^{t+1} c_0}\Big[\theta(g^{t+1}k_0)^{\theta-1}(\gamma^{t+1}h_0)^{1-\theta} + (1-\delta)\Big] \qquad \text{(Euler equation)}$$

$$g^t c_0 + g^t i_0 = (g^t k_0)^\theta (\gamma^t h_0)^{1-\theta} \qquad \text{(resource constaint)}$$

$$\eta g^{t+1} k_0 = (1-\delta) g^t k_0 + g^t i_0 \qquad \text{(LoM of capital)}$$

Here population does matter again, because any invested capital will be distributed over η workers next period. Reducing the g^t:

$$[h_t] : B = \frac{1}{c_0}\Big(\frac{g^\theta \gamma^{1-\theta}}{g}\Big)^t k_0^\theta (1-\theta) h_0^{-\theta} \qquad \text{(consumption-leisure trade-off)}$$

$$[k_{t+1}] : \frac{1}{c_0} = \beta \frac{1}{g c_0}\Big(\frac{\gamma}{g}\Big)^{(1-\theta)(t+1)}\Big[\theta k_0^{\theta-1} h_0^{1-\theta} + (1-\delta)\Big] \qquad \text{(Euler equation)}$$

$$c_0 + i_0 = (k_0)^\theta ((\gamma/g)^t h_0)^{1-\theta} \qquad \text{(resource constaint)}$$

$$\eta g k_0 = (1-\delta) k_0 + i_0 \qquad \text{(LoM of capital)}$$

Now observe that the f.o.c. for h_t has a constant on the LHS, meaning the RHS must be constant too. This is only feasible if $\gamma = g$. Same for the resource constraint.

Together, this system (i) confirms our conjecture and (ii) allows us to solve the system analytically. There are some tricks to doing it. One way is the following:

(1) Simplify the Euler equation $[k_{t+1}]$:

$$1 = \beta \frac{1}{\gamma}\Big[\theta\Big(\frac{k_0}{h_0}\Big)^{\theta-1} + (1-\delta)\Big]$$

This gives us the capital/hours worked ratio at time $t = 0$ in terms of primitive parameters only:

$$\frac{k_0}{h_0} = \Big(\frac{\frac{\gamma}{\beta} - (1-\delta)}{\theta}\Big)^{\frac{1}{\theta-1}}$$

(2) Divide both sides of the LoM of capital by k_0:

$$\frac{i_0}{k_0} = \eta\gamma - (1-\delta)$$

Express as ratio in terms of hours worked (so we can solve for h_0 at the end andplug back in):

$$\frac{i_0}{h_0} = \frac{i_0}{k_0}\frac{k_0}{h_0} = (\eta\gamma - (1-\delta))\Big(\frac{\frac{\gamma}{\beta} - (1-\delta)}{\theta}\Big)^{\frac{1}{\theta-1}}$$

(3) Get the consumption ratio from the resource constraint:

$$\frac{c_0}{h_0} + \frac{i_0}{h_0} = \Big(\frac{k_0}{h_0}\Big)^{\theta} \Longrightarrow \frac{c_0}{h_0} = \Big(\frac{k_0}{h_0}\Big)^{\theta} - (\eta\gamma - (1-\delta))\Big(\frac{\frac{\gamma}{\beta} - (1-\delta)}{\theta}\Big)^{\frac{1}{\theta-1}}$$

(4) Use the consumption-leisure choice to back out the level of h_0:

$$c_0 = \frac{\Big(\frac{k_0}{h_0}\Big)^{\theta}(1-\theta)}{B}$$

and therefore

$$h_0 = \frac{h_0}{c_0}c_0$$

(c) **Show how one can transform this into a stationary (no growth) dynamic programming problem by implementing a change of variables. Show that this leads to the same balanced growth path as obtained in part (b).**

Now that we know for sure at what rate each variable growth on the BGP, we can define their detrended version and be sure to get a version without growth in steady state. Define $\widetilde{x}_t = \frac{x_t}{g_x^t}$, e.g. $\widetilde{k} = \frac{k_t}{\gamma^t}$. For simplicity of notation we shall keep h_t as it is.

We will go through the maximisation problem and rewrite each variable in terms of their detrended counterpart:

$$\widetilde{x}_t = \frac{x_t}{g_x^t} \Longrightarrow x_t = \widetilde{x}_t \gamma^t$$

For some this is algebraically simple; for the objective function, it is less obvious:

$$\beta^t N_t \big[\log(c_t) - Bh_t\big] = \beta^t N_t \big[\log(\widetilde{c}_t) + t\log(\gamma) - Bh_t\big]$$

Or the law of motion:

$$N_{t+1}\widetilde{k}_{t+1}\gamma^{t+1} = (1-\delta)N_t\widetilde{k}_t\gamma^t + N_t\widetilde{i}_t\gamma^t$$

$$N_{t+1}\widetilde{k}_{t+1}\gamma = (1-\delta)N_t\widetilde{k}_t + N_t\widetilde{i}_t$$

The sequential optimization programme can thus be rewritten as:

$$\begin{aligned}
\max_{\{\widetilde{c}_t,\widetilde{h}_t,\widetilde{i}_t,\widetilde{k}'_{t+1}\}} \sum_{t=0}^{\infty} \beta^t N_t \big[log(\widetilde{c}_t) + t\log(\gamma) - Bh_t\big] \quad s.t.\ \forall t:& \\
N_t(\widetilde{c}_t + \widetilde{i}_t) = (N_t\widetilde{k}_t)^\theta (N_t\widetilde{h}_t)^{1-\theta}, \quad \gamma \geq 1,\ \ 0<\theta<1& \\
N_{t+1}\widetilde{k}_{t+1}\gamma = (1-\delta)N_t\widetilde{k}_t + N_t\widetilde{i}_t, \quad 0<\delta<1& \\
k_0 > 0 \text{ given}& \\
N_{t+1} = \eta N_t \quad \eta \geq 1&
\end{aligned}$$

To solve it, proceed with the usual steps. Simplifying it by dropping constants, multipliers etc. in the objective function, and substituting $N_t = \eta^t N_0$:

$$\begin{aligned}
\max_{\{\widetilde{c}_t,\widetilde{h}_t,\widetilde{i}_t,\widetilde{k}'_{t+1}\}} \sum_{t=0}^{\infty} (\beta\eta)^t \big[log(\widetilde{c}_t) - Bh_t\big] \quad s.t.\ \forall t:& \\
\widetilde{c}_t + \widetilde{i}_t = \widetilde{k}_t^\theta \widetilde{h}_t^{1-\theta}, \quad \gamma \geq 1,\ \ 0<\theta<1& \\
\eta\gamma\widetilde{k}_{t+1} = (1-\delta)\widetilde{k}_t + \widetilde{i}_t, \quad 0<\delta<1& \\
k_0 > 0 \text{ given}&
\end{aligned}$$

This is basically the same optimisation problem as before. As a reminder from David's class, here it is solved recursively step-by-step:

$$\begin{aligned}
V(\widetilde{k}) = \max_{c,h,i,k'} N_t \log(\widetilde{c}) - Bh + (\beta\eta)V(\widetilde{k}') \quad s.t.& \\
\widetilde{c} + \widetilde{i} = \widetilde{k}^\theta h^{1-\theta}& \\
\eta\gamma\widetilde{k}' = (1-\delta)\widetilde{k} + \widetilde{i}&
\end{aligned}$$

The Lagrangian:

$$\mathcal{L} = \log(\widetilde{c}) - Bh + (\beta\eta)V(\widetilde{k'}) + \lambda\Big[\widetilde{k}^{\theta}h^{1-\theta} + (1-\delta)\widetilde{k} - \widetilde{c} - \eta\gamma\widetilde{k'}\Big]$$

First-order conditions:

$$\begin{aligned}
[c] &: \frac{1}{\widetilde{c}} = \lambda \\
[h] &: B = \lambda\widetilde{k}^{\theta}(1-\theta)h^{-\theta} \\
[k'] &: (\beta\eta)\frac{\partial\widetilde{k'}}{\partial V}(\widetilde{k'}) = \lambda\eta\gamma \\
[EnvThm] &: \frac{\partial\widetilde{k}}{\partial V}(\widetilde{k}) = \lambda\theta\widetilde{k}^{\theta-1}h^{1-\theta} + (1-\delta) \\
&\implies \frac{\partial\widetilde{k'}}{\partial V}(\widetilde{k'}) = \lambda'\theta\widetilde{k'}^{\theta-1}h'^{1-\theta} + (1-\delta)
\end{aligned}$$

Plugging into each other:

$$\begin{aligned}
[h] &: B = \frac{1}{\widetilde{c}}\Big(\frac{\widetilde{k}}{h}\Big)^{\theta}(1-\theta) \\
[k'] &\ (\beta\eta)\Big[\frac{1}{\widetilde{c'}}\theta\Big(\frac{\widetilde{k'}}{h'}\Big)^{\theta-1} + (1-\delta)\Big] = \frac{1}{\widetilde{c}}\eta\gamma
\end{aligned}$$

Rewrite a bit and combine with constraints:

$$\begin{aligned}
[h] &: B = \frac{1}{\widetilde{c}}\Big(\frac{\widetilde{k}}{h}\Big)^{\theta}(1-\theta) && \text{(consumption-leisure tradeoff)} \\
[k'] &: \frac{1}{\widetilde{c}} = \frac{\beta}{\gamma}\Big[\frac{1}{\widetilde{c'}}\theta\Big(\frac{\widetilde{k'}}{h'}\Big)^{\theta-1} + (1-\delta)\Big] && \text{(Euler equation)} \\
&\widetilde{c} + \widetilde{i} = \widetilde{k}^{\theta}h^{1-\theta} && \text{(resource constraint)} \\
&\eta\gamma\widetilde{k'} = (1-\delta)\widetilde{k} + \widetilde{i} && \text{(law of motion for capital)}
\end{aligned}$$

In steady state, $\widetilde{c'} = c$, $\widetilde{k'} = \widetilde{k}$ etc.:

$$\begin{aligned}
[h] &: B = \frac{1}{\widetilde{c}}\Big(\frac{\widetilde{k}}{h}\Big)^{\theta}(1-\theta) && \text{(consumption-leisure tradeoff)} \\
[k'] &: \frac{1}{\widetilde{c}} = \frac{\beta}{\gamma}\Big[\frac{1}{\widetilde{c}}\theta\Big(\frac{\widetilde{k}}{h}\Big)^{\theta-1} + (1-\delta)\Big] && \text{(Euler equation)} \\
&\widetilde{c} + \widetilde{i} = \widetilde{k}^{\theta}h^{1-\theta} && \text{(resource constraint)} \\
&\eta\gamma\widetilde{k} = (1-\delta)\widetilde{k} + \widetilde{i} && \text{(law of motion for capital)}
\end{aligned}$$

These are literally the same equations in part (b). This system is solved by $\widetilde{c} = c_0$, etc. The steady state of this detrended system is constant, whereas the steady state of the original, non-stationary system is a balanced growth path.
$x_t = \widetilde{x}\gamma^t \implies c_t = \widetilde{c}_t\gamma^t = c_0\gamma^t$ etc.

(d) **Calibrate an annual version of this economy to the following features of the U.S. post-war economy. The number of features should equal the number of parameters to be calibrated.**

- **(i) The average annual growth rate of real output per capita is 2 percent.**
- **(ii) The average annual growth rate of the population is 1 percent.**
- **(iii) The average fraction of total income that is paid to owners of capital is 0.35.**
- **(iv) The average investment-to-output ratio is 0.15**
- **(v) The average capital-to-output ratio is 2.5**
- **(vi) Individuals work 24 hours per capita on average.**

We are asked to find values for the unknown parameters $\beta, B, \theta, \delta, \gamma, \eta$. Calibration is – loosely said – doing some sort of minimum distance estimation, but without bothering with standard errors.[1]

(i) We know that the growth rate of output $y_t = c_t + i_t = f(k_t, h_t)$ increases at rate γ, so calibrate $\gamma = 1.02$.

(ii) The growth rate of population was modelled via the parameter η, so $\eta = 1.01$.

(iii) The capital share is given by the Cobb-Douglas parameter $\theta = 0.35$. With more general production functions, the share is defined as $\frac{r_t k_t}{y_t}$, and one can invoke that in a competitive equilibrium $r_t = \frac{\partial k_t}{\partial f}(k_t, h_t)$, using this for calibration of any parameters.

(iv) We have that

$$\begin{aligned}
\frac{i_t}{y_t} \overset{!}{=} 0.15 = \frac{i_t}{k_t}\frac{k_t}{y_t} &= \frac{i_0}{k_0}\frac{k_0}{y_0} \\
&= \Big(\eta\gamma - (1-\delta)\Big)\frac{k_0}{k_0^\theta h_0^{1-\theta}} \\
&= \Big(\eta\gamma - (1-\delta)\Big)\Big(\frac{k_0}{h_0}\Big)^{1-\theta} \\
&= \Big(\eta\gamma - (1-\delta)\Big)\Big(\frac{\frac{\gamma}{\beta} - (1-\delta)}{\theta}\Big)^{-1}
\end{aligned}$$

[1] Well, until this one came out: Cocci, Plagborg-Møller (2024)

This expression has the still-unknowns β and δ. Use more moments for calibration.

(v) We have that

$$\frac{k_t}{y_t} \stackrel{!}{=} 2.5 = \Big(\frac{\frac{\gamma}{\beta} - (1-\delta)}{\theta}\Big)^{-1}$$

Going back one step, we can now pin down δ:

$$\begin{aligned} \frac{i_t}{y_t} \stackrel{!}{=} 0.15 &= \Big(\eta\gamma - (1-\delta)\Big)\frac{k_t}{y_t} \\ &= \Big(1.01 \cdot 1.02 - 1 + \delta\Big)2.5 \\ \implies \delta &= \frac{0.15}{2.5} + 1 - 1.01 \cdot 1.02 \approx 0.03 \end{aligned}$$

With δ pinned down:

$$\begin{aligned} 2.5 &= \Big(\frac{\frac{\gamma}{\beta} - (1-\delta)}{\theta}\Big)^{-1} \\ &= \Big(\frac{\frac{1.02}{\beta} - (1-0.03)}{0.35}\Big)^{-1} \\ \implies \beta &= \frac{2.02}{\Big(\frac{0.35}{2.5} + (1-0.03)\Big)} \approx 0.92 \end{aligned}$$

(vi) We have the expression for the level of $h_0 = h_t$, which frankly is just a terrible expression, or we can use the resource constraint, factor it in terms of $\frac{i}{y}$ (which is a moment that is given), and plug that into the consumption-leisure tradeoff:

$$\begin{aligned} c_t + i_t = y_t = k_t^\theta h_t^{1-\theta} &\implies c_t = y_t(1 - \frac{i_t}{y_t}) \\ B = \frac{1}{c_t}(1-\theta)\Big(\frac{k_t}{h_t}\Big)^\theta &= \frac{1}{c_t}(1-\theta)h_t^{-1}\underbrace{k_t^\theta h_t^{1-\theta}}_{=y_t} \\ = \frac{(1-\theta)}{h_t(1-\frac{i_t}{y_t})} &= \frac{1-0.35}{24 \cdot (1-0.15)} \approx 0.32 \end{aligned}$$

Question 2: Recursive Competitive Equilibrium

For each of the following economies, do the following:
(i) Specify the dynamic program that would be solved by a social planner in the economy
(ii) Define a recursive competitive equilibrium for the economy

(a) **The are measure one of identical households with preferences given by $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ where $u(\cdot, \cdot)$ is continuous, increasing, concave and continuously differentiable in both arguments. Variables c_t and l_t represent consumption and leisure. There is a constant-returns-to-scale technology $F(K_t, N_t)$ to produce output, where K_t is the capital input and N_t is the labor input. Households are endowed with one unit of time that can be allocated to work, n_t, or leisure l_t. They purchase output from the firm and use it for consumption or as capital in the following period. Capital depreciates at rate δ each period. Households are endowed with k_0 units of capital in period 0.**

(i) **Social Planner's Problem:**
Recursive:

$$
\begin{aligned}
V(k) = \max_{c,l,n,i,k'} \; & u(c,l) + \beta V(k') \\
s.t. \quad & n + l = 1 && \text{(time use constraint)} \\
& k' = (1-\delta)k + i && \text{(law of motion for capital)} \\
& c + i = f(k,n) && \text{(resource constraint)} \\
& c, n, l \geq 0, k' \geq 0 && \text{(boundary conditions)}
\end{aligned}
$$

This is a very verbose way, where each constraint symbolizes an intuitive aspect of the model assumption. We can write it in a more concise way by plugging in some equations into another:

$$
\begin{aligned}
V(k) = & \max_{c,n,k'} u(c, 1-n) + \beta V(k') \\
s.t. \quad & c + k' = f(k,n) + (1-\delta)k && \text{(resource constraint + LoM for capital)} \\
& c \geq 0, k' \geq 0 && \text{(boundary conditions)}
\end{aligned}
$$

For the next paths, I will provide the solution as a more boiled-down version.
For comparison, the **sequential problem** looks like this:

$$
\begin{aligned}
\max_{\{c_t, l_t, n_t, k_{t+1}, i_t\}_{t=0}^{\infty}} \; & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \quad s.t. \forall t \\
& n_t + l_t = 1 && \text{(time use constraint)} \\
& k_{t+1} = (1-\delta)k_t + i_t && \text{(law of motion for capital)} \\
& c_t + i_t = f(k_t, n_t) && \text{(resource constraint)} \\
& k_0 = \overline{k_0} && \text{(initial value)} \\
& c_t \geq 0, 0 \leq n \leq 1, k_{t+1} \geq 0 && \text{(boundary conditions)}
\end{aligned}
$$

which can also be boiled down to fewer equations.
Question: *Why does the recursive problem not feature the initial value?*
Answer: *In the sequential problem, we have to solve for the entire optimal sequences of consumption, leisure, etc. These optimal sequences depend explicitly on* k_0*. In contrast, the recursive problem solves for a mapping: The solution to is not a sequence, but a so-called "value function"* $V(k)$ *along with "policy functions":* $c(k), n(k), k'(k)$*. The key here is that we transform the problem into the state space, and we can drop time indices. It does not matter for the optimal actions* $c(k), n(k), k'(k)$ *what* t *is, only what the state* k *is. With these on hand, one can recover the optimal sequence for any arbitrary* k_0*. Policy functions are very helpful when computing transition paths of the system.*

(ii) **Recursive competitive equilibrium:**
Consider the recursive problem of the household:

$$\begin{aligned} V(k,K) = &\max_{c,n,k'} u(c, 1-n) + \beta V(k', K') \\ s.t. \quad & c + k' = w(K)n + r(K)n + (1-\delta)k && \text{(budget constraint)} \\ & K' = G(K) && \text{(perceived law of motion for aggregate capital)} \\ & c \geq 0, 0 \leq n \leq 1, k' \geq 0 && \text{(boundary conditions)} \end{aligned}$$

k denotes the household's capital stock, and K is the aggregate capital stock.
Consider the (static) problem of the firm:

$$\max_{k^d, n^d} = f(k^d, n^d) - r(K)k^d - w(K)n^d$$

An interior solution to the firm programme exists due to the assumption on $f(\cdot,\cdot)$*.*

A recursive competitive equilibrium consists of:

(1) A household value function $V(k,K)$ and its policy functions $c(k,K), n(k,K), k'(k,K)$
(2) Decision rules for the firm $k^d(K), n^d(K)$ (or how we would call it: factor demand)
(3) Price functions: $w(K), r(K)$
(4) Households' perceived law of motion for aggregate capital $K' = G(K)$

such that:

(i) Given (3) and (4), households solve their recursive problem, yielding (1)
(ii) Given (3), firms solve their static profit problem, yielding (2)
(iii) Markets clear:

- Goods: $c(K, K) + (k'(K, K) - (1-\delta)K) = f(k^d(K), n^d(K))$
- Factor 1 (capital): $k^d = K$
- Factor 2 (labour): $n^d = n(K, K)$

(iv) Perceived laws of motion are correct for households whose idiosyncratic state variables coincide with aggregate ones: $G(K) = k'(K, K)$

Again, for reference the **sequential competitive equlibrium**: First, households take as given the sequence of prices for goods, wage, and capital rental rate $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and optimise their utility objective:

$$\max_{c_t, l_t, n_t, k_{t+1}, i_t} \sum_{t=0}^{\infty} u(c_t, l_t) \quad s.t.$$

$$c_t + i_t = w_t n_t \ \forall t \qquad \text{(budget constraint)}$$

$$n_t + l_t = 1 \ \forall t \qquad \text{(time use constraint)}$$

$$k_{t+1} = (1-\delta)k_t + i_t \forall t \qquad \text{(law of motion for capital)}$$

$$k_0 = \overline{k_0} \qquad \text{(initial value)}$$

Second, firms in all periods t take the prices as given $(w_t, r_t)_{t=0}^{\infty}$ and optimise their profit objective:

$$\max_{K_t, N_t} F(K_t, N_t) - w_t N_t - r_t K_t$$

Lastly, Market clearing conditions hold:

$$c_t + i_t = F(K_t, N_t) \qquad \text{(consumption good market)}$$

$$n_t = N_t \qquad \text{(labour market)}$$

$$k_t = K_t \qquad \text{(capital rental market)}$$

(b) **The same economy in Part A except that utility depends not only on current consumption and leisure, but also on consumption and leisure from the previous period. That is, the period utility function is** $u(c_t, c_{t-1}, l_t, l_{t-1})$.

(i) **Social Planner's Problem:**

For the Planner, the only thing that changes is the utility function of the household. All other (physical) constraints are the same.

$$V(k, c_{-1}, l_{-1}) = \max_{c,l,n,i,k'} u(c, c_{-1}, l, l_{-1}) + \beta V(k', c, l)$$

$$s.t. \quad n + l = 1 \qquad \text{(time use constraint)}$$

$$k' = (1-\delta)k + i \qquad \text{(law of motion for capital)}$$

$$c + i = f(k, n) \qquad \text{(resource constraint)}$$

$$c, n, l \geq 0, k' \geq 0 \qquad \text{(boundary conditions)}$$

(ii) **Recursive competitive equilibrium:**

Consider the recursive problem of the household:

$$
\begin{aligned}
V(k, K, c_{-1}, C_{-1}, l_{-1}, L_{-1}) = \max_{c,l,n,k'} \; & u(c, c_{-1}, l, l_{-1}) + \beta V(k', K', c, C, l, L) \\
s.t. \quad & c + k' = w(K, C_{-1}, L_{-1})n + r(K, C_{-1}, L_{-1})k + (1-\delta)k \\
& \qquad \text{(budget constraint)} \\
& n + l = 1 \qquad \text{(time use constraint)} \\
& (K', C, L) = G(K, C_{-1}, L_{-1}) \\
& \qquad \text{(perceived law of motion for state variables)} \\
& c \geq 0, 0 \leq n \leq 1, k' \geq 0 \qquad \text{(boundary conditions)}
\end{aligned}
$$

k denotes the household's capital stock, and K is the aggregate capital stock.
Consider the (static) problem of the firm:

$$\max_{k^d,n^d} = f(k^d, n^d) - r(K)k^d - w(K)n^d$$

An interior solution to the firm programme exists due to the assumption on $f(\cdot,\cdot)$.

A recursive competitive equilibrium consists of:

(1) A household value function $V(k, K, c_{-1}, C_{-1}, l_{-1}, L_{-1})$ and its policy functions
$c(k, K, c_{-1}, C_{-1}, l_{-1}, L_{-1}), l(k, K, c_{-1}, C_{-1}, l_{-1}, L_{-1})$,
$n(k, K, c_{-1}, C_{-1}, l_{-1}, L_{-1}), k'(k, K, c_{-1}, C_{-1}, l_{-1}, L_{-1})$

(2) Decision rules for the firm $k^d(K), n^d(K)$ (or how we would call it: factor demand)

(3) Price functions: $w(K), r(K)$

(4) Households' perceived laws of motion: $(K', C, L) = G(K, C_{-1}, L_{-1})$

such that:

(i) Given (3) and (4), households solve their recursive problem, yielding (1)

(ii) Given (3), firms solve their static profit problem, yielding (2)

(iii) Markets clear:

- Goods: $c(K, K, C_{-1}, C_{-1}, L_{-1}, L_{-1}) + (k'(K, K, C_{-1}, C_{-1}, L_{-1}, L_{-1}) - (1-\delta)K) = f(k^d(K), n^d(K))$
- Factor 1 (capital): $k^d(K) = K$
- Factor 2 (labour): $n^d(K) = n(K, K, C_{-1}, C_{-1}, L_{-1}, L_{-1})$

(iv) Perceived laws of motion are correct for households whose idiosyncratic state variables coincide with aggregate ones:

$$G(K, C_{-1}, L_{-1}) = \begin{pmatrix} k'(K, K, C_{-1}, C_{-1}, L_{-1}, L_{-1}) \\ c(K, K, C_{-1}, C_{-1}, L_{-1}, L_{-1}) \\ l(K, K, C_{-1}, C_{-1}, L_{-1}, L_{-1}) \end{pmatrix}$$

For both (i) and (ii), just swap out the utility function to $u(c_t, c_{t-1}, l_t, l_{t-1})$. Note that c_{t-1} and l_{t-1} become a state variable to track, which will affect the mapping of the value and policy functions.

(c) **The same economy as in Part A except that there are two firms that operate constant-returns technologies. One produces new capital goods from labor and existing capital supplied by the households. The other produces consumption goods from these same factors of production. Denote the technologies operating in sector i by $f^i(k_i, n_i)$, $i = 1, 2$. Households have one unit of time that can be allocated to leisure, labor in sector 1, $n_{1,t}$, and sector 2, $n_{2,t}$. In addition, households accumulate productive capital that can be allocated to either sector.**

(i) **Social Planner's Problem:**

$$V(k) = \max_{c,n_1,n_2,k_1,k_2,k'} u(c, 1 - n_1 - n_2) + \beta V(k') s.t.$$

$$k_1 + k_2 = k \qquad \text{(capital use constraint)}$$

$$c = f^1(k_1, n_1) \qquad \text{(output good constraint)}$$

$$k' = (1 - \delta)k + f^2(k_2, n_2) \quad \text{(capital good constraint + LoM for capital)}$$

$$n_1 \geq 0, n_1, n_2 \geq 0, k_1, k_2 \geq 0$$

Some bondary conditions are not needed anymore, such as $c \geq 0$ is given due to assuming $f(\cdot, \cdot) \geq 0$.

For reference, this is the problem in **sequence form**. Let's start again with the verbose version:

$$\max_{\{c_t,l_t,n_t^1,n_t^2,k_t^1,k_t^2,k_{t+1},i_t\}_{t+0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \quad s.t.$$

$$\begin{aligned}
& n_t^1 + n_t^2 + l_t = 1 && \text{(time use constraint)} \\
& k_t^1 + k_t^2 = k_t && \text{(capital use constraint)} \\
& k_{t+1} = (1-\delta)k_t + i_t && \text{(law of motion for capital)} \\
& c_t = f^1(k_t^1, n_t^1) && \text{(resource constraint for consumption goods)} \\
& i_t = f^2(k_t^2, n_t^2) && \text{(resource constraint for capital)} \\
& k_0 = \overline{k_0} && \text{(initial value)}
\end{aligned}$$

or boiled down:

$$\max_{\{c_t,n_t^1,n_t^2,k_t^1,k_t^2,k_{t+1},i_t\}_{t+0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t^1 - n_t^2) \quad s.t.$$

$$\begin{aligned}
& k_t^1 + k_t^2 = k_t && \text{(capital use constraint)} \\
& k_{t+1} = (1-\delta)k_t + f^2(k_t^2, n_t^2) && \text{(law of motion for capital)} \\
& c_t = f^1(k_t^1, n_t^1) && \text{(resource constraint for consumption goods)} \\
& k_0 = \overline{k_0} && \text{(initial value)}
\end{aligned}$$

(ii) **Recursive competitive equilibrium:**
Consider the recursive problem of the household:

$$\begin{aligned}
V(k,K) = & \max_{c,n,k'} u(c, 1-n) + \beta V(k', K') \\
s.t. \quad & c + q(K)k' = w(K)n + r(K)k + (1-\delta)q(K)k && \text{(budget constraint)} \\
& K' = G(K) && \text{(perceived law of motion for aggregate capital)} \\
& c \geq 0, 0 \leq n \leq 1, k' \geq 0 && \text{(boundary conditions)}
\end{aligned}$$

k denotes the household's capital stock, and K is the aggregate capital stock.
Consider the (static) problem of the consumption good firm:

$$\max_{k^{d,1},n^{d,1}} = f^1(k^{d,1}, n^{d,1}) - r(K)k^{d,1} - w(K)n^{d,1}$$

Consider the (static) problem of the capital good firm:

$$\max_{k^{d,2},n^{d,2}} = q(K)f^2(k^{d,2}, n^{d,2}) - r(K)k^{d,2} - w(K)n^{d,2}$$

Capital goods have a relative price of $g(K)$, in units of consumption goods (numeraire).

Define the recursive competitive equilibrium to consist of:

(1) A household value function $V(k, K)$ and its policy functions $c(k, K), n(k, K), k'(k, K)$

(2) Consumption good firm decision functions (or "factor demands") $k^{d,1}(K), n^{d,1}(K)$

(3) Capital good firm decision functions (or "factor demands") $k^{d,2}(K), n^{d,2}(K)$

(4) Price functions $w(K)$, $r(K)$, $q(K)$

(5) Households' perceived law of motion for aggregate capital $K' = G(K)$

such that

(i) Given (4) and (5), households solve their recursive problem, yielding (1)

(ii) Given (4), consumption good firms solve their profit problem, yielding (2)

(iii) Given (4), capital good firms solve their prodit problem, yielding (3)

(iv) Markets clear:

- Consumption goods: $c(K, K) = f^1(k^{d,1}(K), n^{d,1}(K))$
- Capital goods: $k'(K, K) - (1 - \delta)K = f^2(k^{d,2}(K), n^{d,2}(K))$
- Factor 1 (capital): $k^{d,1}(K) + k^{d,2}(K) = K$
- Factor 2 (labour): $n^{d,1}(K) + n^{d,2}(K) = n(K, K)$

(v) Perceived laws of motion are correct for households whose idiosyncratic state variables coincide with aggregate ones: $G(K) = k'(K, K)$

Question: *Here we have made the implicit assumption that the labour market in sectors* 1 *and* 2 *is common, with one single wage* $w(K)$. *Now someone says: What if the labour markets were separated? How would you answer?*

One suggested answer: *One could certainly write the model with segmented labour markets. It will be more equations and variables to track: Households would have to make a decision on where to work. However, in equilibrium, wages will be equal. Otherwise, the household would only work in one single sector. Intuitively, one can see that this would not be an equilibrium: The household demands the other good, but that other sector would have zero output, failing market clearance ... Writing this down is actually very annoying, and one art of the economist is to make our lives easier – so let's just stick to the common labour market from the get-go :)*

Question 3: Postwar Growth in the United States, Germany, and Japan

This problem explores how a simple neoclassical growth model fares in explaining the post-war growth experiences of the US, Germany, and Japan. We will do so using the following model. Consider an economy with measure one of identical households whose preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

where c_t is consumption and $\sigma > 0$. Capital depreciates at rate δ, where $0 < \delta < 1$. The initial capital stock is k_0. The production technology is given by

$$Y_t = K_t^{\theta}(\gamma^t H_t)^{1-\theta}$$

where K_t and H_t are the capital and labor inputs, and $\gamma \geq 1$ represents efficiency growth. The household rents its capital to the market at rate r_t, and rents its labor services at wage w_t. For simplicity, households supply labor inelastically: $H = 1$.

(a) **Express the social planner's problem for this economy as a recursive problem.**

$$\begin{aligned}
V(k, A) = \max_{\{c,i,k'\}_{t=0}^{\infty}} & \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V(k', A') \quad s.t. \\
& c + i = k^{\theta}(Ah_t)^{1-\theta} \\
& k' = (1-\delta)k + i \\
& A' = \gamma A \\
& c, k' \geq 0
\end{aligned}$$

(b) **Solve the planner's problem and derive the optimality conditions.**

$$\begin{aligned}
\mathcal{L} &= \frac{c^{1-\sigma}}{1-\sigma} + \beta V(k', \gamma A) + \lambda \Big[k^{\theta}(Ah)^(1-\theta) + (1-\delta)k - c - k'\Big] \\
[c] &: c^{-\sigma} = \lambda \\
[k'] &: \beta \frac{\partial}{\partial k'} V(k', A') = \lambda \\
[EnvThm] &: \frac{\partial}{\partial k} V(k, A) = \lambda \Big[\theta \Big(\frac{k}{Ah}\Big)^{\theta-1} + (1-\delta)\Big] \\
[\lambda] &: c + k' = k^{\theta}(Ah)^{1-\theta} + (1-\delta)k
\end{aligned}$$

Reduced to:

$$\begin{aligned}
& c^{-\sigma} = \beta(c')^{-\sigma}\Big[\theta(k')^{\theta-1}(A'h')^{1-\theta} + 1 - \delta\Big] && \text{(Euler equation)} \\
& c + k' = k^{\theta}(Ah)^{1-\theta} + (1-\delta)k && \text{(resource constraint)} \\
& A' = \gamma A && \text{(law of motion for labour productivity)}
\end{aligned}$$

Plugging in the law of motion for labour productivity, and $h = 1$ constant:

$$c^{-\sigma} = \beta(c')^{-\sigma}\Big[\theta(k')^{\theta-1}(\gamma A)^{1-\theta} + 1 - \delta\Big] \qquad \text{(Euler equation)}$$
$$c + k' = k^{\theta}(A)^{1-\theta} + (1-\delta)k \qquad \text{(resource constraint)}$$

(c) **We want to use the tools learned in the last quarter. Writing code for a value function with two state variables can be awkward (additional loops or using Kronecker products), so we will use a trick. Given the insight from Question 1(c), stationarize the original problem by appropriately detrending the variables. You should then not have any state variable for "technology", or any γ^t-terms. Argue that you can use a recursive method to computationally solve our original model, even when not on the balanced growth path.**
Hint: The Euler equation holds on any equilibrium path, even if not on the balanced growth path.

Change-of-variables in the original problem:

$$\widetilde{c}_t = \frac{c_t}{\gamma^t}, \widetilde{k}_t = \frac{k_t}{\gamma_t}, \widetilde{i}_t = \frac{i_t}{\gamma_t}$$

makes the problem:

$$\max_{\{\widetilde{c}_t, \widetilde{i}_t, \widetilde{k}_{t+1}\}} \sum_{t=0}^{\infty} \beta^t (\gamma^{1-\sigma})^t \frac{\widetilde{c}_t^{1-\sigma}}{1-\sigma} \quad s.t.$$
$$\widetilde{c}_t + \widetilde{i}_t = \widetilde{k}_t^{\theta}$$
$$\gamma \widetilde{k}_{t+1} = (1-\delta)\widetilde{k}_t + \widetilde{i}_t$$

So the detrended version can be characterised as the same problem but with:

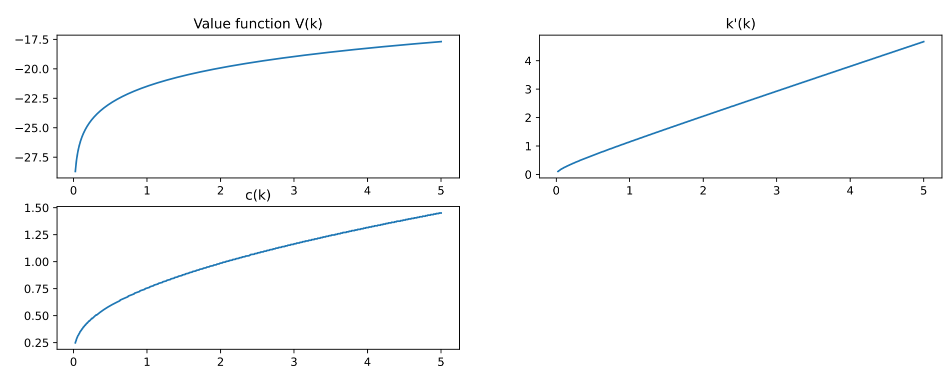
- New discount factor $\hat{\beta} = \beta\gamma^{1-\sigma}$
- New law of motion for capital $k_{t+1} = \frac{(1-\delta)k_t + i_t}{\gamma}$

We will proceed with the detrended version for now, and omit the tilde-notation, at least until later when we need to undo the detrending again.
Blackwell's Theorem tells us that if the Bellman operator is a contraction, value function iteration is guaranteed to work. We can either check for the requirements explicitly (i.e. confirming sufficient conditions), or we can just let the code run and see if our chosen metric for V does indeed go to zero at an acceptable rate.

Important: It only holds on the balanced growth path that $c_{t+1} = \gamma c_t$. If there was a disturbance that brings us away from the BGP, we may have different transition dynamics, but whatever path the economy takes, it still is an equilibrium. Here, we take the entire problem and simply rewrite it in terms of detrended variables, so it is an innocuous change-of-variable. We **do not** rely on the fact that γ is the BGP growth rate. We could technically write it in terms of whatever detrend we want. We happen to choose γ because it detrends the system into a stationary one, and it works for no growth rate other than the BGP growth rate(s). With a stationary system, we can use existing toolkits to solve it. One way you have learned is value function iteration.

Description of the value function iteration (VFI) algorithm: For the code, we can recycle any VFI code skeleton for one state variable. We have to make two adjustments: First, change the effective discount factor to $\hat{\beta}$. Second, we have to incorporate the new law of motion. Since VFI finds the maximum $u(c) + \beta V(k')$ at each iteration step, we can write it as a choice over the optimal $k'(k)$ – given that, the optimal choice of $c(k)$ is determined by the resource contraint. One way to code up the backward iteration step is to put the discretised values of k into different rows, and discretised potential choices of k' into different columns, and maximisation happens rowwise over all columns. The proposed instantaenous utility depends on our choice of k': $u(c(k)) = u\Big(c(k'(k))\Big)$, where $c(k'(k)) = k^\theta - i$ as per the resource constraint. Now restate the new law of motion in terms of i and plug in: $i_t = \gamma k'(k) - (1-\delta)k$, so $c(k'(k)) = k^\theta + (1-\delta)k - \gamma k'(k)$. So our code could be structured as such: For each row k, compute over all the columns the value $c(k'(k))$, and then $V^{prop}(k'(k)) = u(c(k'(k))) + \beta V(k'(k))$. Over all these values among the columns, choose which $k'(k)$ makes you most well off. Save the maximising $k'(k)$ as temporary policy function and $V(k) = V^{prop}(k'(k))$ as temporary value function. Do this over all rows k, and iterate the temporary value functions until they converge element-wise for all k.

The value function $V(k)$ along with the policy functions $c(k), k'(k)$ are plotted here:



We seem to converge to a steady-state, which is reassuring.

(d) **Using value function iteration, numerically solve for the planner's policy functions assuming that $\delta = 0.1$, $\beta = 0.96$, $\theta = 0.3$, $\gamma = 1.01$, $\theta = 2$, and $k_0 = 0.5$. This will represent the United States. Report the competitive equilibrium allocation for the first 30 periods after the initial period, to represent the years 1950 through 1979. On four different plots (or 2x2 subplots in Matlab), plot the following over these 40 periods:**

  (i) Log GDP per capita

  (ii) The rental rate of capital (the marginal product of capital)

  (iii) The wage rate (the marginal product of labour)

  (iv) The investment-output ratio

For the transition path, we can use the policy functions and iterate forward: Take the initial value of the state variable, k_0, and get $c_1 = c(k_0)$ and $k_1 = k'(k_0)$. To compute the other objects, use the model restrictions: $y_t = k_t^\theta$, $mpk_t = \theta k_t^{\theta-1}$, $i_t = y_t - c_t$, etc. Afterwards, iterate again: $c_2 = c(k_1)$, $k_2 = k'(k_1)$, and so on. Plotting this transition path in the stationarised system:

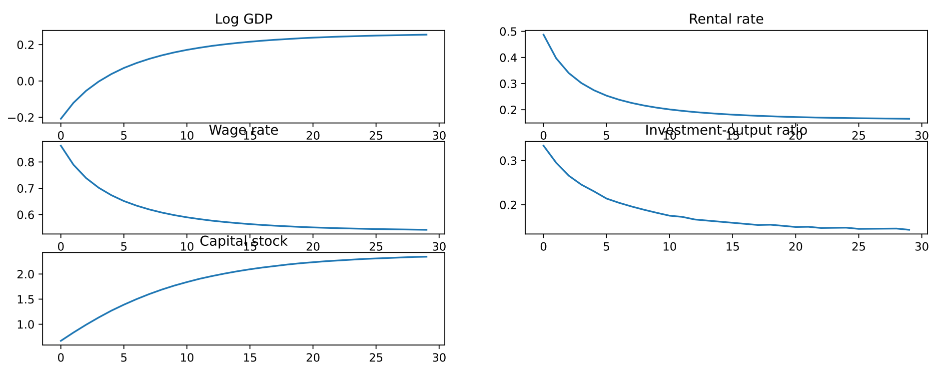


Figure 1: $k_0 = 0.5$, stationarised system

We see that

As a last step, we have to undo the detrending in order to plot the path in the original, nonstationary system. Now using the tilde-notation again:

- Log GDP per capita:

$$\begin{aligned} \widetilde{y}_t &= \widetilde{k}_t^{\theta} \\ \Longrightarrow \frac{y_t}{\gamma^t} &= \widetilde{k}_t^{\theta} \\ \Longrightarrow \log(y_t) - t\log(\gamma) &= \theta \log(\widetilde{k}_t) \\ \Longrightarrow \log(y_t) &= \theta \log(\widetilde{k}_t) + t\log(\gamma) \end{aligned}$$

- Marginal product of capital:

$$\begin{aligned} mpk_t &= \frac{\partial}{\partial k_t} k_t^{\theta}(\gamma^t h_t)^{1-\theta} = \theta k_t^{\theta-1}(\gamma^t h_t)^{1-\theta} \\ &\overset{h_t=1}{=} \theta \Big(\frac{k_t}{\gamma^t}\Big)^{\theta-1} = \theta \Big(\frac{\widetilde{k}_t\gamma^t}{\gamma^t}\Big)^{\theta-1} = \theta \widetilde{k}_t^{\theta-1} \end{aligned}$$

- Marginal product of labour:

$$\begin{aligned} mpl_t &= \frac{\partial}{\partial h_t} k_t^{\theta}(\gamma^t h_t)^{1-\theta} = (1-\theta) k_t^{\theta}(\gamma^t h_t)^{-\theta}\gamma^t \\ &\overset{h_t=1}{=} (1-\theta)\Big(\frac{k_t}{\gamma^t}\Big)^{\theta}\gamma^t = (1-\theta)\Big(\frac{\widetilde{k}_t\gamma^t}{\gamma^t}\Big)^{\theta}\gamma^t = (1-\theta)\widetilde{k}_t^{\theta}\gamma^t \end{aligned}$$

- Investment-output ratio:

$$\frac{i_t}{y_t} = \frac{y_t - c_t}{y_t} = 1 - \frac{c_t}{c_t} = 1 - \frac{\widetilde{c}_t}{\widetilde{y}_t}$$

Plotting that:

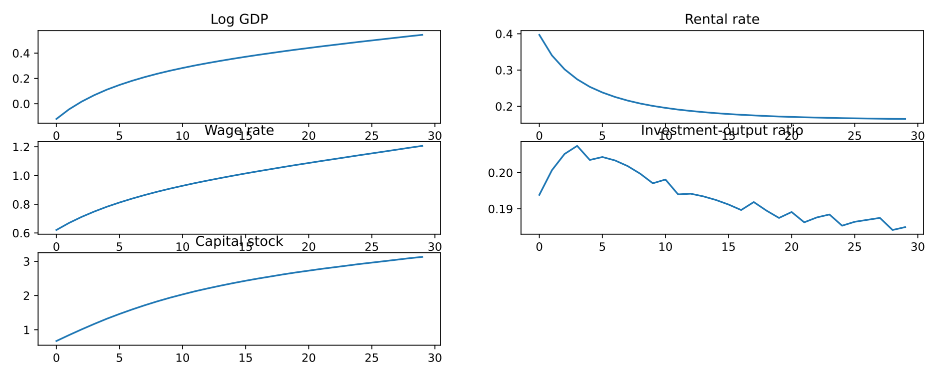


Figure 2: $k_0 = 0.5$, system with long-term growth

Also reassuringly, we seem to converge to something that looks like balanced growth, and wage and rental rates converge to a constant (in the BGP, their growth rate is constantly zero). We see that the investment-output ratio is behaving a bit erratically. We can see in the plot below that these seem to be numerical issues, which turn out to be largely fine.

(e) Now, using your code for part (b), repeat the graphic by assuming that $k_0 = 0.1$.

We change the starting value of the simulation to $k_0 = 0.1$. Note that we **do not** need to change the policy function, since it maps to the optimal choice from any k!

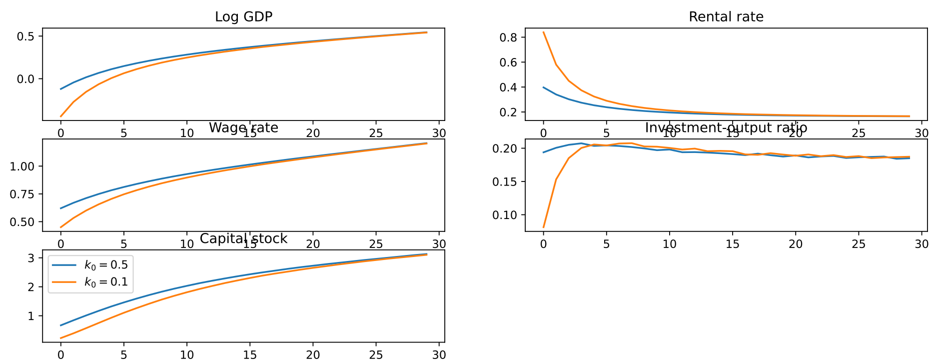


Figure 3: system with long-term growth, $k_0 = 0.5$ vs $k_0 = 0.1$

Most strikingly, we see that even though there is long-run growth, the economy starting at the lower initial value manages to catch-up to the one starting with much more capital. After a while, both seem to be on the balanced growth path. The take-away from this model is: In the long-run, initial capital allocations do not matter.

(f) **Using the data accompanying this Problem Set on canvas, plot postwar growth in the US, Germany, and Japan in one plot. Express it in terms of log GDP, and compare it to the transition path from the model simulations. Does our model do a good job at replicating broad patterns?**

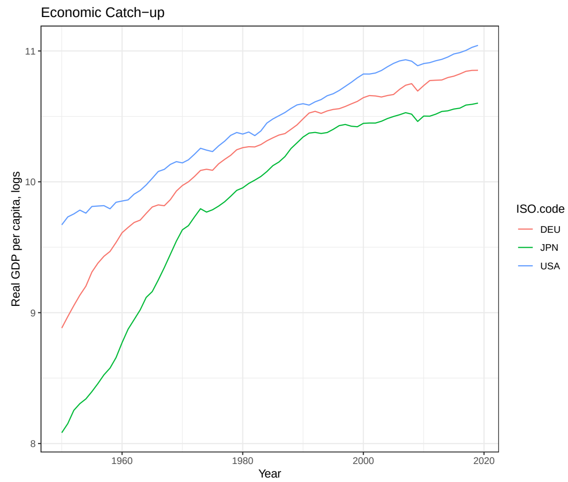


Figure 4: We have a similar catch-up, both not fully in levels

Figure 5: A similar pattern emerges with the capital stock, and the level difference becomes more apparent.

Overall, the shape is very similar: Even though the US seems to grow at a constant rate, Germany and Japan manage to catch up almost all the way. A gap still remains, though all countries seem to be on a balanced growth path afterwards. We see though that the catch-up is not quite as fast as our model predicts, even if we shifted things "mentally" so that Germany and Japan completely catch up.

(g) **Can you use intuition from our growth model to explain the Japanese/German**

catch-up after World War 2?

According to the growth model, after destruction of capital, the returns to capital are extremely high due to concavity of the production function. This means that investing some capital will yield tremendous amounts of additional output, whereas the US with its larger stockpile of capital sees only modest returns. Even though consumption is also lower in Germany and Japan just after the war, the marginal product of capital being high implies high interest rates, thus providing the household with an incentive to save (via the Euler equation). Still, as our utility is also concave, neither population is wiling to starve just to quickly accumulate capital. Therefore, investment rates are high but not "too high" at the start. As these high returns are realized, capital stock is gradually built up. With this build-up come smaller marginal products, slowly decreasing the incentive to save. At some point, the economies catch up. The fact that we have exogenous growth does not change this fact. Even though the US experiences persistent growth, improving technology (i) also affects existing capital in Germany and Japan, and (ii) further increases their incentive to save, due to higher future marginal products.

(h) **Consider the following modification to the model. Household preferences are now given by:**

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t - \bar{c})^{1-\sigma}}{1-\sigma}$$

Interpret the parameter $\bar{c}$. Which aspect of the model fit might be improved by it? Describe briefly using economic intuition how you expect this modification work.

One interpretation is to call $\bar{c}$ a "subsistence parameter". We can see that the household wants at least $\bar{c}$, say a basic amount of nutrition for survival. This makes the utility function more concave at lower levels of consumption, and increases household marginal utility of consumption there. We can intuit with the Euler equation, where we would put more emphasis on consumption today than in the future, and therefore tilt away from investment relative to the baseline specification. This could help us fix the overly fast catch-up in the model. It does not help us with the persistent long-run differences. One possible fix for that is to assume a country-specific level of productivity, with the US just being innately more efficient. However, this would not explain another observation that Japan ends up with higher capital stocks than Germany, but has lower output. We could find another fix for that.

Generally one has to be careful with modeling too much "towards data". This model is intended to explain what seems to be a tendency for countries to catch up to a frontier, no matter their initial capital stock. It is not intended to match every other little detail we may find in the data. If we introduce more degrees to freedom, our model will obviously match the data better, but it's unclear if we are overfitting. Further, it may detract from the core mechanism or even obfuscate it by introducing too many concurrent changes. Lastly, some features may fit the data better, but don't actually explain it. For example, why should the US be innately more efficient than other countries? Introducing a fixed parameter fits the data, but yields almost no deeper insight.