# ECON 2010C — Problem Set 2 – Suggested Solutions

November 20, 2024

## Question 1: Capital Taxation in the Growth Mode

**Consider an economy with measure one of identical households whose preferences are given by**

$$\sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \alpha \log(1 - n_t)\right]$$

**where $c_t \geq 0$ and $0 < n_t < 1$ for all $t$ and where each household is endowed with some initial capital level $k_0$. Let the production function be $F(K_t, N_t) = AK_t^\theta N_t^{1-\theta}$ for $A > 0$, and suppose that the technology is operated by a representative firm that behaves perfectly competitively. Suppose that the government levies a tax $\tau^k$ on capital income each period, satisfying $0 < \tau^k < 1$, and spends the proceeds on "foreign aid" which does not affect the economy in any way.**

(a) **Define a recursive competitive equilibrium for this economy.**

Household problem:

$$\begin{aligned} V(k, K) = \max_{c,n,k'} & \log(c) + \alpha \log(1 - n) + \beta V(k', K') \\ \text{s.t} \quad & c + k' = (1 - \delta)k + w(K)n + (1 - \tau^k)r(K)k \\ & c \geq 0, \quad n \geq 0, \quad k' \geq 0 \end{aligned}$$

Firm problem:

$$\max_{K^d, N^d} A(K^d)^\theta (N^d)^{1-\theta} - w(K)N^d - r(K)K^d$$

A recursive competitive equilibrium is given by:

(a) The value function $V(K, k)$ and policy functions $c(K, k)$, $n(K, k)$ and $k'(K, k)$ for the household

(b) Firm decision rules $N^d(K), K^d(K)$

(c) Price functions $r(K)$ and $w(K)$

(d) Perceived law of motion $\hat{G}(K)$

such that

(a) Given (3) and (4), (1) solves the household's problem.

(b) Given (3), (2) solves the firm's problem.

(d) Markets clear $N^d(K) = n(K, K)$, and $K^d(K) = K$.

(e) Perceptions are correct $\hat{G}(K) = k'(K, K)$

(b) Characterize the recursive competitive equilibrium. Solve for the steady-state values of the return on capital, the capital-labor ratio, the wage rate, consumption per unit of labor, and labor input.

First recognise that distortionary taxation leads to a failure of the First Welfare Theorem. We therefore have to solve for the competitive equilibrium and can not resort to using the planner's solution.
Start with the household problem:

$$\begin{aligned}\mathcal{L} =& \log(c) + \alpha \log(1-n) + \beta V(K', k') + \\ & \lambda\Big[(1-\delta)k + w(K)n + (1-\tau^k)r(K)k - c - k'\Big]\end{aligned}$$

First-order conditions:

$$\begin{aligned}[c]: &\quad \frac{1}{c} = \lambda \\ [n]: &\quad \alpha\frac{1}{1-n} = \lambda w(K) \\ [k']: &\quad \beta\frac{\partial}{\partial k'}V(k', K') = \lambda \\ [EnvThm]: &\, \frac{\partial}{\partial k}V(k, K) = \lambda\Big[(1-\delta) + (1-\tau^k)r(K)\Big] \\ [\lambda]: &\, c + k' = (1-\delta)k + w(K)n + (1-\tau^k)r(K)k\end{aligned}$$

Rewrite:

$$\alpha\frac{1}{1-n} = \frac{1}{c}w(K) \qquad \text{(consumption-leisure choice)}$$

$$\frac{1}{c} = \beta\frac{1}{c'}\Big[1-\delta + (1-\tau^k)r(K')\Big] \qquad \text{(Euler equation)}$$

$$c + k' = (1-\delta)k + w(K)n + (1-\tau^k)r(K)k \qquad \text{(budget constraint)}$$

Firm problem yields first-order conditions:

$$[N^d] : A(1-\theta)(K^d)^\theta (N^d)^{-\theta} = w(K)$$
$$[K^d] : A\theta(K^d)^{\theta-1}(N^d)^{1-\theta} = r(K)$$

which can be written as

$$A(1-\theta)\Big(\frac{K^d}{N^d}\Big)^\theta = w(K)$$
$$A\theta\Big(\frac{K^d}{N^d}\Big)^{\theta-1} = r(K)$$

For the steady state, impose additional restrictions that $c' = c$, $k'(K, K) = K$ etc. From the Euler equation we pin down the steady state interest rate:

$$r = \frac{1}{1-\tau^k}\Big(\frac{1}{\beta} - 1 + \delta\Big)$$

Using $r$ and the factor demand for capital we pin down the SS capital-labour ratio:

$$\mathcal{K} := \frac{K}{N} = \left(\frac{A\theta}{\frac{1}{1-\tau^k}(\frac{1}{\beta} - 1 + \delta)}\right)^{\frac{1}{1-\theta}} = (1-\tau^k)^{\frac{1}{1-\theta}} \left(\frac{A\theta}{(\frac{1}{\beta} - 1 + \delta)}\right)^{\frac{1}{1-\theta}}$$

Using the SS capital-labour ratio, we pin down SS wages:

$$w = A(1-\theta)(\mathcal{K})^\theta$$

From the household budget constraint at the steady state we get consumption per unit of labour:

$$\begin{aligned}
c &= ((1-\tau^k)r - \delta)K + wN \\
\implies \frac{c}{N} &= ((1-\tau^k)r - \delta)\frac{K}{N} + w \\
&= ((1-\tau^k)\theta A(\frac{K}{N})^{\theta-1} - \delta)\frac{K}{N} + A(1-\theta)(\frac{K}{N})^\theta \\
&= (1-\tau^k)\theta A(\frac{K}{N})^\theta - \delta\frac{K}{N} + A(1-\theta)(\frac{K}{N})^\theta \\
&= \theta A(\frac{K}{N})^\theta - \tau^k\theta A(\frac{K}{N})^\theta - \delta\frac{K}{N} + A(\frac{K}{N})^\theta - \theta A(\frac{K}{N})^\theta \\
&= A(\frac{K}{N})^\theta - \delta\frac{K}{N} - \tau^k\theta A(\frac{K}{N})^\theta \qquad (1) \\
&= A\mathcal{K}^\theta - \delta\mathcal{K} - \tau^k\theta A\mathcal{K}^\theta \qquad (2)
\end{aligned}$$

From the labor-leisure condition we pin down $N$ at the steady state:

$$\begin{aligned} \alpha c &= (1-N)w \\ \alpha \frac{c}{N} &= (\frac{1}{N} - 1)w \\ \frac{\alpha}{w}\frac{c}{N} + 1 &= \frac{1}{N} \\ N &= \frac{1}{\frac{\alpha}{w}\frac{c}{N} + 1} \end{aligned}$$

(c) **Now specify the problem of a social planner who has to spend as much each period on foreign aid as in the competitive equilibrium allocation. You can let $g_t$ represent the expenditure on foreign aid that the planner must make in period $t$. Solve for the planner's steady-state values of the marginal product of capital, the capital-labor ratio, the marginal product of labor, consumption per unit of labor, and labor input.**

A social planner has to "spend" $g_t = \tau^k rK$ every period on foreign aid, where $r, K$ are the values from the competitive equilibrium. Taking this into consideration, let's form the social planner's solution:

$$\begin{aligned} V(K) = \max_{C,N,K'} & \log C + \alpha \log(1-N) + \beta V(K') \\ \text{s.t} \quad & C + K' + g = AK^\theta N^{1-\theta} + (1-\delta)K, \\ & C \geq 0, \quad N \geq 0, \quad K' \geq 0 \end{aligned}$$

where $g$ is treated as given, and happens to take on the values above.

Lagrangian of the social planner's problem:

$$L(\mu, C, N, K') = \log C + \alpha \log(1-N) + \beta V(K') + \mu(AK^\theta N^{1-\theta} + (1-\delta)K - C - K' - g)$$

Let's take the FOCs of the social planner's solutions:

$$[C] : \frac{1}{C} = \mu \tag{3}$$

$$[N] : \alpha \frac{1}{1-N} = \mu A(1-\theta)(\frac{K}{N})^\theta \tag{4}$$

$$[K'] : \beta V'(K') = \mu \tag{5}$$

$$[EnvThm] : \frac{\partial V(K)}{\partial k} = \mu(\theta A(\frac{K}{N})^{\theta-1} + 1 - \delta) \tag{6}$$

Using equations (3) and (4) we have the consumption-leisure condition for the social planner:

$$\alpha\frac{1}{1-N}=\frac{1}{C}A(1-\theta)(\frac{K}{N})^{\theta} \tag{7}$$

Using the equations (5) and (6), we have derive the Euler Equation:

$$\beta\frac{1}{C'}(\theta A(\frac{K'}{N'})^{\theta-1}+1-\delta)=\frac{1}{C} \tag{8}$$

And the aggregate resource constraint is:

$$C+K'+g=AK^{\theta}N^{1-\theta}+(1-\delta)K$$

Now let's characterize the steady state of the planner.

First, from the Euler Equation (8) we pin down the capital-labor ratio for the social planner solution:

$$\mathcal{K}_{SP}=\frac{K_{SP}}{N_{SP}}=\left(\frac{\theta A}{\frac{1}{\beta}-1+\delta}\right)^{\frac{1}{1-\theta}} \tag{9}$$

and

$$MPK_{SP}=\theta A(\frac{K_{SP}}{N_{SP}})^{\theta-1}=\frac{1}{\beta}-1+\delta$$
$$MPL_{SP}=A(1-\theta)(\frac{K_{SP}}{N_{SP}})^{\theta}=A(1-\theta)\left(\frac{\theta A}{\frac{1}{\beta}+1-\delta}\right)^{\frac{\theta}{1-\theta}}$$

From the Social planner's resource constraint:

$$C+g=AK^{\theta}N^{1-\theta}-\delta K$$
$$\frac{C}{N}=A(\frac{K}{N})^{\theta}-\delta\frac{K}{N}-\frac{g}{N}$$
$$\frac{C}{N}=A(\frac{K}{N})^{\theta}-\delta\frac{K}{N}-\frac{\tau_K r_{CE}K_{CE}}{N_{SP}}$$
$$\frac{C}{N}=A(\frac{K}{N})^{\theta}-\delta\frac{K}{N}-\tau^K r_{CE}\frac{K_{CE}}{N_{CE}}\frac{N_{CE}}{N_{SP}}$$

where we need to differentiate between the steady state in the competitive equilibrium (CE), which determines the magnitude of $g$, and the social planner (SP).

From the consumption-leisure condition (7),

$$\alpha C = A(1-\theta)(\frac{K}{N})^{\theta}(1-N)$$
$$\alpha\frac{C}{N} = A(1-\theta)(\frac{K}{N})^{\theta}(\frac{1}{N}-1)$$
$$\alpha\frac{C}{N} = A(1-\theta)(\frac{K}{N})^{\theta}(\frac{1}{N}-1)$$
$$N_{SP} = \frac{1}{\frac{\alpha}{A(1-\theta)(\frac{K}{N})^{\theta}}\frac{C}{N}+1}$$

Here the $\frac{C}{N}$-ratio differs from the competitive equilibrium.

(d) **How does the competitive allocation differ from the planner's allocation? Provide intuition for your answer making reference to the differences in your answers to (b) and (c).**

Let's compare CE and SP allocations.
First note that , for $\tau^k \in (0,1)$

$$MPK_{CE} = r_{CE} = \frac{1}{1-\tau^k}(\frac{1}{\beta}-1+\delta) > MPK_{SP} = \frac{1}{\beta}-1+\delta$$
$$\frac{K_{CE}}{N_{CE}} = (1-\tau^k)^{\frac{1}{1-\theta}}\left(\frac{A\theta}{(\frac{1}{\beta}-1+\delta)}\right)^{\frac{1}{1-\theta}} < \frac{K_{SP}}{N_{SP}} = \left(\frac{\theta A}{\frac{1}{\beta}+1-\delta}\right)^{\frac{1}{1-\theta}}$$

meaning that capital-labour ratio in the CE is lower that that of the social planner's economy. This implies that marginal product of labour is higher in the planner's solution compared to the CE:

$$MPL_{CE} = w_{CE} = A(1-\theta)(\frac{K_{CE}}{N_{CE}})^{\theta} < MPL_{SP} = A(1-\theta)(\frac{K_{SP}}{N_{SP}})^{\theta}$$

Now let's compare consumption per capita in these two cases. From the previous parts b, c we have:

$$\frac{C_{SP}}{N_{SP}} = A(\frac{K_{SP}}{N_{SP}})^{\theta} - \delta\frac{K_{SP}}{N_{SP}} - \tau^K r_{CE}\frac{K_{CE}}{N_{CE}}\frac{N_{CE}}{N_{SP}} \quad (10)$$
$$\frac{C_{CE}}{N_{CE}} = A(\frac{K_{CE}}{N_{CE}})^{\theta} - \delta\frac{K_{CE}}{N_{CE}} - \tau^k r_{CE}\frac{K_{CE}}{N_{CE}} \quad (11)$$

Let's define a function $g(K/N)$ s.t

$$g(\frac{K}{N}) = A(\frac{K}{N})^{\theta} - \delta\frac{K}{N}$$

Note that g is a concave and attains its maximum when $g'(K/N) = A\theta(\frac{K}{N})^{\theta-1} - \delta = 0$, i.e when MPK is equal to depreciation. However, note that in steady state, the marginal product of capital is larger than the rate of depreciation since the household is impatient: $MPK_{CE} > MPK_{SP} > \frac{1}{\beta} - 1 + \delta > \delta$. Since MPK is decreasing in $\frac{K}{N}$, it must be the case that $\frac{K_{CE}}{N_{CE}} > \frac{K_{SP}}{N_SP} > \frac{K_{max}}{N_{max}}$ where $\frac{K_{max}}{N_{max}}$ is the capital labor ratio that maximizes the value of g. But since g is concave, and capital labor ratio in both CE and SP cases are lower than the maximizer, it must be that g is increasing in the capital labor ratio near both CE and SP capital-labor ratio. Thus, we have that

$$g(\frac{K_{SP}}{N_{SP}}) > g(\frac{K_{CE}}{N_{CE}})$$

We can rewrite equations 10 and 11 as

$$\frac{C_{SP}}{N_{SP}} = g(\frac{K_{SP}}{N_{SP}}) - \tau^K r_{CE}\frac{K_{CE}}{N_{CE}}\frac{N_{CE}}{N_{SP}} \tag{12}$$

$$\frac{C_{CE}}{N_{CE}} = g(\frac{K_{CE}}{N_{CE}}) - \tau^k r_{CE}\frac{K_{CE}}{N_{CE}} \tag{13}$$

Now whether $\frac{C_{SP}}{N_{SP}} \lesseqgtr \frac{C_{CE}}{N_{CE}}$ depends on the ratio $\frac{N_{CE}}{N_{SP}}$
We know that

$$N_{CE} = \frac{1}{\frac{\alpha}{A(1-\theta)(\frac{K_{CE}}{N_{CE}})^\theta}\frac{C_{CE}}{N_{CE}} + 1}$$

$$N_{SP} = \frac{1}{\frac{\alpha}{A(1-\theta)(\frac{K_{SP}}{N_{SP}})^\theta}\frac{C_{SP}}{N_{SP}} + 1}$$

Taking the ratio and plugging in consumption-per capita expressions we have

$$\frac{N_{CE}}{N_{SP}} = \frac{\frac{\alpha}{A(1-\theta)(\frac{K_{SP}}{N_{SP}})^\theta}\frac{C_{SP}}{N_{SP}} + 1}{\frac{\alpha}{A(1-\theta)(\frac{K_{CE}}{N_{CE}})^\theta}\frac{C_{CE}}{N_{CE}} + 1} = \frac{\alpha\frac{C_{SP}}{N_{SP}} + A(1-\theta)(\frac{K_{SP}}{N_{SP}})^\theta}{\alpha\frac{C_{CE}}{N_{CE}} + A(1-\theta)(\frac{K_{CE}}{N_{CE}})^\theta}\frac{(\frac{K_{CE}}{N_{CE}})^\theta}{(\frac{K_{SP}}{N_{SP}})^\theta} =$$

$$\frac{A(\frac{K_{SP}}{N_{SP}})^\theta - \delta\frac{K_{SP}}{N_{SP}} - \tau^K r_{CE}\frac{K_{CE}}{N_{CE}}\frac{N_{CE}}{N_{SP}} + \frac{1}{\alpha}A(1-\theta)(\frac{K_{SP}}{N_{SP}})^\theta}{A(\frac{K_{CE}}{N_{CE}})^\theta - \delta\frac{K_{CE}}{N_{CE}} - \tau^k r_{CE}\frac{K_{CE}}{N_{CE}} + \frac{1}{\alpha}A(1-\theta)(\frac{K_{CE}}{N_{CE}})^\theta}\frac{(\frac{K_{CE}}{N_{CE}})^\theta}{(\frac{K_{SP}}{N_{SP}})^\theta} =$$

$$\frac{A - \delta\frac{K_{SP}}{N_{SP}}^{1-\theta} - \tau^K r_{CE}\frac{K_{CE}}{N_{CE}}(\frac{K_{SP}}{N_{SP}})^{-\theta}\frac{N_{CE}}{N_{SP}} + \frac{1}{\alpha}A(1-\theta)}{A - \delta\frac{K_{CE}}{N_{CE}}^{1-\theta} - \tau^k r_{CE}\frac{K_{CE}}{N_{CE}}^{1-\theta} + \frac{1}{\alpha}A(1-\theta)}$$

Now we need to solve for $\frac{N_{CE}}{N_{SP}}$

$$\frac{N_{CE}}{N_{SP}}(A-\delta\frac{K_{CE}}{N_{CE}}^{1-\theta}-\tau^k r_{CE}\frac{K_{CE}}{N_{CE}}^{1-\theta}+\frac{1}{\alpha}A(1-\theta)) = A-\delta\frac{K_{SP}}{N_{SP}}^{1-\theta}-\tau^K r_{CE}\frac{K_{CE}}{N_{CE}}(\frac{K_{SP}}{N_{SP}})^{-\theta}\frac{N_{CE}}{N_{SP}}+\frac{1}{\alpha}A$$

$$\frac{N_{CE}}{N_{SP}}(A-\delta\frac{K_{CE}}{N_{CE}}^{1-\theta}-\tau^k r_{CE}\frac{K_{CE}}{N_{CE}}^{1-\theta}+\frac{1}{\alpha}A(1-\theta)+\tau^K r_{CE}\frac{K_{CE}}{N_{CE}}(\frac{K_{SP}}{N_{SP}})^{-\theta}) = A-\delta\frac{K_{SP}}{N_{SP}}^{1-\theta}+\frac{1}{\alpha}A(1-$$

$$\frac{N_{CE}}{N_{SP}} = \frac{A-\delta\frac{K_{SP}}{N_{SP}}^{1-\theta}+\frac{1}{\alpha}A(1-\theta)}{A-\delta\frac{K_{CE}}{N_{CE}}^{1-\theta}+\frac{1}{\alpha}A(1-\theta)-\tau^K r_{CE}\frac{K_{CE}}{N_{CE}}[\frac{K_{CE}}{N_{CE}})^{-\theta}-(\frac{K_{SP}}{N_{SP}})^{-\theta}]}$$

Since $\frac{K_{SP}}{N_{SP}} > \frac{K_{CE}}{N_{CE}}$), $N_{CE} < N_{SP}$, when $\tau_k$ is not too large. Indeed, $N_{CE} < N_{SP}$ when

$$\tau_K < \delta\frac{\frac{K_{SP}}{N_{SP}}^{1-\theta}-\frac{K_{CE}}{N_{CE}}^{1-\theta}}{r_{CE}\frac{K_{CE}}{N_{CE}}[\frac{K_{CE}}{N_{CE}})^{-\theta}-(\frac{K_{SP}}{N_{SP}})^{-\theta}]}$$

As a result, production is lower in CE than in SP economy.

Using 13 and 12 we can see right away that $\frac{C_{CE}}{N_{CE}} < \frac{C_{SP}}{N_{SP}}$. Further, we can compare the relative consumption levels:

$$\frac{C_{CE}}{C_{SP}} = \frac{C_{CE}/N_{CE}}{C_{SP}/N_{SP}}\frac{N_{CE}}{N_{SP}} = \frac{g(\frac{K_{CE}}{N_{CE}})-\tau^k r_{CE}\frac{K_{CE}}{N_{CE}}}{g(\frac{K_{SP}}{N_{SP}})-\tau^K r_{CE}\frac{K_{CE}}{N_{CE}}\frac{N_{CE}}{N_{SP}}}\frac{N_{CE}}{N_{SP}} =$$

$$\frac{g(\frac{K_{CE}}{N_{CE}})-\tau^k r_{CE}\frac{K_{CE}}{N_{CE}}}{g(\frac{K_{SP}}{N_{SP}})\frac{N_{SP}}{N_{CE}}-\tau^K r_{CE}\frac{K_{CE}}{N_{CE}}} =$$

Note that $\frac{C_{CE}}{C_{SP}} < 1$ because $\frac{N_{SP}}{N_{CE}}$ and $g(\frac{K_{SP}}{N_{SP}}) > g(\frac{K_{CE}}{N_{CE}})$ So, the both the level of consumption and consumption per capita are higher in the planner's allocation compared to the competitive equilibrium.

(e) **Let $\alpha$=1, $A$=1, $\theta$=0.33, $\tau_K$ = 0.25, $\beta$=0.96. Start in steady state and consider two scenarios (1) the planner's allocation each period, and (2) the competitive equilibrium allocation with the marginal capital taxation, but the household "magically" gets consumption of c*(1+x) each period rather than c. Compute the consumption equivalent variation, that is the value x, making the households indifferent between (1) and (2). Interpret the value x in this context.**

The consumption equivalent welfare measure finds the percent increase in consumption of the competitive equilibrium required to make the household indifferent between the compensated competitive steady state and the planner's steady state. Since we are in steady state, it suffices to compare instantaneous utilities. Specifically, we solve for $x$ such that:

$$\begin{aligned}
U((1+x)C_{CE}, 1-N_{CE}) &\stackrel{!}{=} U(C_{SP}, 1-N_{SP}) \\
\log((1+x)C_{CE}) + \alpha\log(1-N_{CE}) &= \log(C_{SP}) + \alpha\log(1-N_{SP}) \\
\log(1+x) &= \log(\frac{C_{SP}}{C_{CE}}) + \alpha\log(\frac{1-N_{SP}}{1-N_{CE}}) \\
1+x &= \frac{C_{SP}}{C_{CE}}(\frac{1-N_{SP}}{1-N_{CE}})^{\alpha} \\
x &= \frac{C_{SP}}{C_{CE}}(\frac{1-N_{SP}}{1-N_{CE}})^{\alpha} - 1
\end{aligned}$$

For given parameter values, $x = 4.05\%$, meaning that consumption needs to be increased by 4.05% to match the welfare level in the planner's solution. This is the welfare loss due to the distortionary nature of taxation.

## Question 2: Pollution and Abatement

An economy is populated by measure one of households whose preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t \{\log(C_t) - \alpha P_t^2\},$$

where $0 < \beta < 1$, $C_t \geq 0$ is consumption, $\alpha > 0$ and $P_t \geq 0$ is pollution. The households each supply one unit of labor inelastically. Output is produced using the following production technology:

$$Y_t = A(K_t)^{\theta}(N_t)^{1-\theta}$$

where $0 < \theta < 1$, $A > 0$, and variables $Y_t$, $K_t$ and $N_t$ represent output, capital and labor. Pollution is a by-product of the production process, but it can be abated (reduced) using "pollution abatement equipment," denoted $X_t$. Specifically, the pollution produced at time $t$ is given by

$$P_t = \phi\frac{Y_t}{X_t}$$

where $\phi > 0$. The current period's output can be transformed into consumption, abatement equipment or capital for the following period. All the capital and abatement equipment used in the current period depreciates fully after it is used, and the abatement equipment cannot be stored from one period to the next. The resource constraint of the economy is $Y_t = C_t + X_t + K_{t+1}$, where $K_{t+1}$ is capital to be saved for the following period.

(a) Provide a recursive formulation of the social planner's problem.

$$
\begin{aligned}
V(K) = \max_{C,N,Y,P,K',X} & \log C - \alpha P^2 + \beta V(K') \\
\text{s.t.} \quad & Y = A(K)^\theta N^{1-\theta} \\
& P = \phi \frac{Y}{X} \\
& C + X + K' = Y \\
& N \leq 1 \\
& C \geq 0, \quad N \geq 0, \quad , K' \geq 0
\end{aligned}
$$

(b) **Characterize the solution to the social planner's problem. Provide a brief interpretation of each optimality condition.**

$$
L = \log C - \alpha P^2 + \beta V(K') + \Omega(A(K)^\theta N^{1-\theta} - Y) + \Lambda(Y - C - X - K') + \Pi(P - \phi \frac{Y}{X}) + \Gamma(1 - N)
$$

$$
\begin{aligned}
&[C]: \quad \frac{1}{C} = \Lambda \\
&[N]: \quad \Omega(1-\theta)AK^\theta N^{-\theta} = \Gamma \\
&[Y]: \quad -\Omega + \Lambda - \Pi\phi\frac{1}{X} = 0 \\
&[P]: \quad -2\alpha P + \Pi = 0 \\
&[K']: \quad \beta V'(K') - \Lambda = 0 \\
&[X]: \quad -\Lambda + \Pi\phi\frac{Y}{X^2} = 0 \\
&[EnvThm]: \quad V'(K) = \Omega A \theta K^{\theta-1} N^{1-\theta}
\end{aligned}
$$

Now let's combine all the conditions above:

$$
\Pi = 2\alpha P \tag{14}
$$

$$
\frac{1}{C} = 2\alpha P \phi \frac{Y}{X^2} \tag{15}
$$

$$
\Omega = \frac{1}{C} - 2\alpha P \phi \frac{1}{X} \tag{16}
$$

$$
\beta(\frac{1}{C'} - 2\alpha P \phi \frac{1}{X'}) A \theta K'^{\theta-1} N'^{1-\theta} = \frac{1}{C} \tag{17}
$$

Equation 15 tells us the tradeoff between using resources for "pollution abatement equipment" and consumption, i.e. the marginal utility of consumption should be equal marginal benefit of pollution abatement. The pollution abatement technology is concave, so the more resources are allocated to abatement, the less is its marginal benefit $\phi\frac{Y}{X^2}$, while the pollution function is convex and marginal disutility of pollution is $2\alpha P$.

The equation 17 is the usual intratemporal Euler Equation. There is an extra term $-2\alpha P\phi\frac{1}{X'}$ appearing in the marginal benefit tomorrow due to pollution, because more capital tomorrow leads to more production and hence pollution.

(c) **Now assume that resources are allocated in competitive markets, and that the government regulates that each household purchases $\overline{x}$ units of abatement equipment each period. Formulate the household's problem recursively, and define a recursive competitive equilibrium.**

Household's problem:

$$\begin{aligned} V(K,k) = \max_{c,k'} \log c - \alpha \hat{P}(K)^2 + \beta V(K',k') \\ \text{s.t} \quad c + k' + \bar{x} = w(K)n + r(K)k \\ c \geq 0, \quad n = 1, \quad k' \geq 0 \\ \text{[perceived laws of motion]} : \quad K' = \hat{G}(K) \\ \hat{P}(K) \end{aligned}$$

Firm's problem:

$$\max_{K^d,N^d} AK^{d^\theta} N^d 1 - \theta - w(K)N^d - r(K)K^d$$

Recursive competitive equilibrium is:

(a) Value function V(K, k) and decision rules for the HH $c(K,k)$ $k'(K,k)$

(b) decision rules of the firm $N^d(K)$ and $K^d$

(c) price functions $w(K)$ and $r(K)$

(d) Perceived law of motion for capital $K'(K,k) = \hat{G}(K)$ and for pollution $\hat{P}(K)$

such that

(a) Given 3 and 4, 1 solves the HH problem

(b) Given 3, 2 solves the firm's problem

(c) Markets clear: $K^d = K$, $N^d(K) = 1$

(d) Perceptions are correct: $\hat{G}(K) = k'(K,K)$ and $\hat{P}(K) = \phi\frac{AK^\theta}{\bar{x}}$

(d) **Providing a set of conditions that characterize the recursive competitive equilibrium.**

$$L(c, k', \lambda) = \log c - \alpha \hat{P}(K)^2 + \beta V(K', k') + \lambda(w(K) + r(K)k - c - k' - \bar{x})$$

FOCs:

$$\begin{aligned} [c]: &\quad \frac{1}{c} = \lambda \\ [k']: &\quad \beta V_2(K', k') = \lambda \\ [EnvThm]: &\quad V_2(K, k) = \lambda r(K) \end{aligned}$$

Combining these we have the Euler Equation:

$$\beta \frac{1}{c'} r(K') = \frac{1}{C}$$

Firm's FOC are:

$$\begin{aligned} w(K) &= A(1-\theta)(K^d)^\theta (N^d)^{-\theta} \\ r(K) &= A\theta (K^d)^{\theta-1} (N^d)^{1-\theta} \end{aligned}$$

Note that by market clearing $N^d = 1$.

(e) **What are the steady-state values of capital, output and pollution in the recursive competitive equilibrium? How does the equilibrium capital stock depend on $\bar{x}$? Provide a brief intuition for your answer.**

At the steady state $r = \frac{1}{\beta}$, $N = 1$ and

$$K = (A\theta\beta)^{\frac{1}{1-\theta}}$$

and

$$Y = A(A\theta\beta)^{\frac{\theta}{1-\theta}}$$

and pollution is

$$P = \phi \frac{Y}{\bar{x}} = \phi \frac{A(A\theta\beta)^{\frac{\theta}{1-\theta}}}{\bar{x}}$$

Equilibrium capital stock does not depend on spending on the pollution abatement technology, because $\bar{x}$ enters the budget constraint of the HH as a lump-sum, non-distortionary tax. Thus, households do not internalize their impact on $\bar{x}$, and thus on pollution.

## Question 3: "Baumol's Disease" in Services

An economy is populated by measure one of households that have preferences over manufacturing goods, $m_t$, and services, $s_t$. Their preferences are:

$$\sum_{t=0}^{\infty} \beta^t \left[\mu m_t^{\frac{\epsilon-1}{\epsilon}} + (1-\mu)(s_t + \bar{s})^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$$

where $\mu$ is a constant satisfying $0 < \mu < 1$, $\bar{s}$ is a positive constant, and $\epsilon$ is the elasticity of substitution between manufacturing and services, satisfying $0 < \epsilon < \infty$. The technology to produce manufacturing output is

$$Y_t^m = A_{m,t} N_{m,t},$$

where $N_{m,t}$ is the labor input in manufacturing, and $A_{m,t}$ represents manufacturing efficiency. The technology to produce services is

$$Y_t^s = A_s N_{s,t},$$

where $N_{s,t}$ is service labor input and $A_s$ is service efficiency. Manufacturing efficiency grows exogenously at rate $\gamma_m$ each period: $A_{m,t+1} = A_{m,t}(1+\gamma_m)$, where $\gamma_m > 0$. Service production efficiency grows exogenously at rate $\gamma_s$ each period: $A_{s,t+1} = A_{s,t}(1+\gamma_s)$, for $\gamma_s > 0$.

Neither $Y_t^s$ nor $Y_t^m$ is storable from one period, and both goods are restricted to be non-negative. Employment in the two sectors must satisfy $N_{m,t} + N_{s,t} = 1$ for all $t$.

(a) **Define the social planner's problem for this economy. Characterize the solution to the planner's problem.**

The SP's problem can be described as a fully static problem and is given by:

$$\begin{aligned}
\max_{C_t^m, C_t^s, N_t^m, N_t^s} \quad & [\mu (C_t^m)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\mu)(C_t^s + \bar{s})^{\frac{\varepsilon-1}{\varepsilon}}]^{\frac{\varepsilon}{\varepsilon-1}} \\
\text{s.t} \quad & Y_t^m = A_{m,t} N_{m,t} \\
& Y_t^s = A_{s,t} N_{s,t} \\
& N_t^s + N_t^m = 1 \\
& A_{m,t+1} = A_{m,t}(1+\gamma_m) \\
& A_{s,t+1} = A_{s,t}(1+\gamma_s) \\
& N_t^m \geq 0, \quad N_t^s \geq 0 \\
& 0 \leq C_t^m \leq Y_t^m, \quad 0 \leq C_t^s \leq Y_t^s
\end{aligned}$$

First, note that at the optimum $C_t^m = Y_t^m$ and $C_t^s = Y_t^s$, so the problem can be written as:

$$\begin{aligned}
&\max_{N_{s,t}} [\mu(A_{m,t}(1-N_{s,t}))^{\frac{\varepsilon-1}{\varepsilon}} + (1-\mu)(A_{s,t}N_{s,t}+\bar{s})^{\frac{\varepsilon-1}{\varepsilon}}]^{\frac{\varepsilon}{\varepsilon-1}} \\
&\text{s.t} \quad A_{m,t+1} = A_{m,t}(1+\gamma_m) \\
&\qquad A_{s,t+1} = A_{s,t}(1+\gamma_s) \\
&\qquad N_{s,t} \geq 0
\end{aligned}$$

$$L(\lambda, N_{s,t}) = [\mu(A_{m,t}(1-N_{s,t}))^{\frac{\varepsilon-1}{\varepsilon}} + (1-\mu)(A_{s,t}N_{s,t}+\bar{s})^{\frac{\varepsilon-1}{\varepsilon}}]^{\frac{\varepsilon}{\varepsilon-1}} + \lambda N_{s,t}$$

The FOC w.r.t. $N_{s,t}$ is

$$\begin{aligned}
\frac{\varepsilon}{\varepsilon-1} C_t^{\frac{1}{\varepsilon}} [-\mu A_{m,t}^{\frac{\varepsilon-1}{\varepsilon}} \frac{\varepsilon-1}{\varepsilon}(1-N_{s,t})^{-\frac{1}{\varepsilon}} + (1-\mu)\frac{\varepsilon-1}{\varepsilon}(A_{s,t}N_{s,t}+\bar{s})^{-\frac{1}{\varepsilon}} A_{s,t}] + \lambda = 0 \\
C_t^{\frac{1}{\varepsilon}} [-\mu A_{m,t}^{\frac{\varepsilon-1}{\varepsilon}} (1-N_{s,t})^{-\frac{1}{\varepsilon}} + (1-\mu)(A_{s,t}N_{s,t}+\bar{s})^{-\frac{1}{\varepsilon}} A_{s,t}] + \lambda = 0
\end{aligned}$$

where $C_t = [\mu(A_{m,t}(1-N_{s,t}))^{\frac{\varepsilon-1}{\varepsilon}} + (1-\mu)(A_{s,t}N_{s,t}+\bar{s})^{\frac{\varepsilon-1}{\varepsilon}}]^{\frac{\varepsilon}{\varepsilon-1}}$.

Complementary-slackness condition tells us that $\lambda N_{s,t} = 0$, i.e either:

(1) $\lambda = 0 \quad \& \quad N_{s,t} > 0$ or

(2) $\lambda > 0 \quad \& \quad N_{s,t} = 0$

(b) **Are both goods consumed in all periods? If so, justify your answer. If not, characterize when both goods are consumed and when just one good is consumed.**

Consider case (1), under which both goods are consumed: $\lambda = 0 \quad \& \quad N_{s,t} > 0$.

$$\begin{aligned}
-\mu A_{m,t}^{\frac{\varepsilon-1}{\varepsilon}} (1-N_{s,t})^{-\frac{1}{\varepsilon}} + (1-\mu)(A_{s,t}N_{s,t}+\bar{s})^{-\frac{1}{\varepsilon}} A_{s,t} = 0 \\
\mu A_{m,t}^{\frac{\varepsilon-1}{\varepsilon}} (1-N_{s,t})^{-\frac{1}{\varepsilon}} = (1-\mu)(A_{s,t}N_{s,t}+\bar{s})^{-\frac{1}{\varepsilon}} A_{s,t}
\end{aligned}$$

and $N_{s,t}$ can be found using the above equation.

Consider case (2), under which only manufacturing good is consumed: $\lambda > 0 \quad \& \quad N_{s,t} = 0$. We then have:

$$
\begin{aligned}
C_t^{\frac{1}{\varepsilon}}[-\mu A_{m,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\mu)(\bar{s})^{-\frac{1}{\varepsilon}} A_{s,t}] &< 0 \\
(1-\mu)(\bar{s})^{-\frac{1}{\varepsilon}} A_{s,t} &< \mu A_{m,t}^{\frac{\varepsilon-1}{\varepsilon}} \\
(1-\mu)(\bar{s})^{-\frac{1}{\varepsilon}} A_{s,0}(1+\gamma_s)^t &< \mu(A_{m,0}(1+\gamma_m)^t))^{\frac{\varepsilon-1}{\varepsilon}} \\
\frac{(A_{m,0}(1+\gamma_m)^t))^{\frac{\varepsilon-1}{\varepsilon}}}{A_{s,0}(1+\gamma_s)^t} &> \frac{1-\mu}{\mu}\bar{s}^{-\frac{1}{\varepsilon}} \\
\frac{\mu}{1-\mu}\bar{s}^{\frac{1}{\varepsilon}} &> \frac{A_{s,0}(1+\gamma_s)^t}{(A_{m,0}(1+\gamma_m)^t))^{\frac{\varepsilon-1}{\varepsilon}}}
\end{aligned}
$$

The above expression gives us the condition under which only manufacturing good is consumed and $N_{m,t} = 1$. Let $\varepsilon > 1$, then if the manufacturing sector grows very fast $\gamma_m > \gamma_s$, then only manufacturing good will be produced, given that $\bar{s}$ is high enough. Another obvious case is when $\mu = 1$, i.e. households only care about manufacturing sector.

(c) **Now suppose that $\gamma_m > \gamma_s$. What happens to the ratio of manufacturing employment to total employment in the long run? (as $t \to \infty$)? Explain the intuition for your answer.**

Similar to class problem, we consider 2 cases:

Case 1 (gross substitutes ): $\varepsilon > 1$: the share of manufacturing employment goes to 1 in the long run.

Case 2 (gross complements): $0 < \varepsilon < 1$: The share of manufacturing employment goes to 0 in the log run.

## Question 4: Structural Change in a Roy Model of Sorting

There are measure one of agents in an economy. Half are "strong" types that have a comparative advantage in agricultural tasks. Half are "smart" types that have a comparative advantage in services. In particular, strong types are endowed with $z$ units of agricultural labor units, and one unit of service labor units, where $z > 1$. Smart types are endowed with one unit of agricultural labor units and $z$ units of service labor units. Each agent can work in exactly one sector, and supplies all of her labor units to that sector.

All agents have the following preferences:

$$\sum_{t=0}^{\infty} \beta^t [\log(c_t^a - \overline{a}) + \log(c_t^s)],$$

where $c_t^a$ is consumption of agriculture goods, and $c_t^s$ is consumption of services, both of which much be non-negative each period. The parameter $\overline{a}$ is a positive constant that represents a subsistence requirement. The production functions for agriculture and services are:

$$Y_t^a = A_t N_t^a \qquad \text{and} \qquad Y_t^s = A_t N_t^s,$$

where $N_t^a$ is the input of agricultural labor units, $N_t^n$ is the input of service labor units, and $A_t$ is an exogenous efficiency level. The initial efficiency level is $A_0 = 1$, and the law of motion for $A_t$ is $A_{t+1} = A_t(1+g)$ where $g > 0$.

Let services be the numeraire good, and define the relative price of agriculture be $p_t^a$. Let the wage per agricultural labor unit be $w_t^a$ and the wage per service labor unit be $w_t^s$. Note that $w_t^a$ needn't equal $w_t^s$ since they are prices for different types of labor inputs. Assume that both technologies are operated by competitive firms with unrestricted entry.

(a) **Write down the problems for strong types and for smart types, taking prices are given, as sequence problems.**

For the household, this problem is static again taken prices and wages as given. We normalise the price of the services sector to be one:

$$\max\ \log(c_t^a - \overline{a}) + \log(c_t^s)\ s.t.\ p_t^a c_t^a + c_t^s = \max\{w_t^a \overline{z}, w_t^s\} =: I^a(w_t^a, w_t^s) \text{ for strong types}$$
$$\max\ \log(c_t^a - \overline{a}) + \log(c_t^s)\ s.t.\ p_t^a c_t^a + c_t^s = \max\{w_t^a, w_t^s \overline{z}\} =: I^s(w_t^a, w_t^s) \text{ for smart types}$$

We can conceptually separate the problem of the household for choosing the sector of work and choosing the bundle of consumption.
Type strong provide labour supply:

$$n(w_t^a, w_t^s) = \begin{cases} (\overline{z}, 0) & \text{if } w_t^a \overline{z} > w_t^s \\ \text{either of the two} & \text{if } w_t^a \overline{z} = w_t^s \\ (0, 1) & \text{if } w_t^a \overline{z} < w_t^s \end{cases}$$

and vice versa for the smart type. So income $I^i(w_t^a, w_t^s)$ is

$$I^i(w_t^a, w_t^s) = \begin{cases} w_t^i \overline{z} & \text{if } w_t^i \overline{z} > w_t^{-i} \\ w_t^i \overline{z} & \text{if } w_t^i \overline{z} = w_t^{-i} \\ w_t^{-i} & \text{if } w_t^i \overline{z} < w_t^{-i} \end{cases}$$

Consumption given some income $I = I^i(w_t^a, w_t^s), i \in \{a, s\}$ is characterised by:

$$\begin{aligned} \frac{1}{c_t^a - \overline{a}} = \frac{p_t^a}{I - p_t^a c_t^a} &\implies p_t^a (c_t^a - \overline{a}) = I - p_t^a c_t^a \\ &\implies 2 p_t^a c_t^a - p_t^a \overline{a} = I \\ &\implies c_t^a = \frac{1}{2} \frac{I}{p_t^a} + \frac{\overline{a}}{2} \end{aligned}$$

and accordingly

$$\begin{aligned} c_t^s &= I - p_t^a c_t^a \quad s.t. c_t^s \geq 0 \\ &= I - \frac{1}{2} - p_t^a \frac{\overline{a}}{2} \quad s.t. c_t^s \geq 0 \\ &= \max\{\frac{1}{2} I - p_t^a \frac{\overline{a}}{2}, 0\} \end{aligned}$$

(b) **Write down the static firms' problems. What must be true of the competitive equilibrium labor prices $w_t^a$ and $w_t^s$ in equilibrium? Briefly explain your answer.**

For any sector $i \in \{a, s\}$:

$$\max_{N_t^i} \; p_t^i A_t N_t^i - w_t^i N_t^i \implies N_t^{i,d} = \begin{cases} \infty & , \text{ if } p_t^i A_t > w_t^i \\ \mathbb{R}_{\geq 0} & , \text{ if } p_t^i A_t = w_t^i \\ 0 & , \text{ if } p_t^i A_t < w_t^i \end{cases}$$

We can immediately rule out the first case for either sector, as that would bust the labour market clearing condition: We have at most $\frac{1}{2}\overline{z} + \frac{1}{2}$ of labour supply in any sector, given our setting of strong and smart types. If it must be that $p_t^a A_t \leq w_t^a$ and simultaneously $A_t \leq w_t^s$.

(c) **Characterize the optimal sorting patterns of the workers in period 0, and the equilibrium relative price of agriculture goods, as a function of $A_0$.**

Everything here is static. So it all depends on the value of $A_t$ relative to $\bar{a}$ and $\bar{z}$. A priori, we distinguish between several cases: (i) All work in agriculture. (ii) All strong types work in agriculture, some smart types too – the rest of thsoe works in services. (iii) All strong types work in agriculture, all smart types work in services. (iv) All smart types work in services, some strong ones too – the rest of those works in agriculture. (v) All types work in services. (vi) Some smart types work in agriculture, some strong ones in services.

First note from household demand that if $c_t^s > 0 \implies c_t^a > 0$. This is caused by the subsistence term $\bar{a}$: Households first want to make sure they don't starve, and only after this requirement is met do they worry about consuming services. This will help us rule out some cases very quickly later.

Now suppose we are in case (ii). That assumption will pin down wages. First compute the respective income given those wages, then consumption, and then we look at market clearing conditions. If both types work in agriculture, that implies that smart types are indifferent between working in either sector, i.e. $I_t^s = w_t^a = w_t^s \bar{z}$. The income for strong types is $I_t^a = w_t^a \bar{z}$. Because there is positive and finite output in both sectors, the firm conditions imply that $w_t^a = p_t^a A_t$ and $w_t^s = A_t$. Demand for agrucultural produce by both types is subsequently

$$\begin{aligned} c_t^{a,a} &= \frac{1}{2}\frac{I_t^a}{p_t^a} + \frac{\bar{a}}{2} \\ &= \frac{1}{2}A_t\bar{z} + \frac{\bar{a}}{2} \\ &= \frac{1}{2}(A_t\bar{z} + \bar{a}) \end{aligned}$$

and

$$\begin{aligned} c_t^{a,s} &= \frac{1}{2}\frac{I_t^s}{p_t^a} + \frac{\bar{a}}{2} \\ &= \frac{1}{2}A_t + \frac{\bar{a}}{2} \\ &= \frac{1}{2}(A_t + \bar{a}) \end{aligned}$$

The number of farmhands needed is:

$$\frac{\frac{1}{2}c_t^{a,a} + \frac{1}{2}c_t^{a,s}}{A_t} = \frac{1}{4}(\bar{z}+1) + \frac{1}{2}\frac{\bar{a}}{A_t} = N_t^a$$

In this case, $I_t^a = w_t^a\bar{z} = p_t^a A_t\bar{z} = p_t^a w_t^a$; the last step following from revealed preference of the smart types $w_t^s\bar{z} = A_t\bar{z} = w_t^a$. This implies $p_t^a = \bar{z}$, and we can confirm that strong types would not work in services: $I_t^a = A_t\bar{z}^2 > A_t$, which would have been their income in services. One may be tempted to do the following sanity check: If there were no subsistence needs ($\bar{a} = 0$), then preference for agriculture and services would be symmetric, and labour demand in $a$ would be $\frac{1}{2}(\frac{1}{2}\bar{z} + \frac{1}{2}\cdot 1)$. This is counter to intuition, since it looks like an inefficient allocation of types across sectors (half-and-half each). However, if there was no subsistence need, we would not be in this case in the first case, as we shall see later.

Now suppose we are in case (i), where all types work in agriculture. The calculation is similar is in case (ii), except that here the wage in services is not pinned down. We know that smart types earn $I_t^s = w_t^a = p_t A_t$, which is what we have been calculating with before. So this case is just the corner case of (ii) where there is exactly zero labour (and output or consumption) in services.

Now suppose we are in a perfectly separated labour market, i.e. case (iii). Then we must have $w_t^a\bar{z} > w_t^s$ from the strong types and $w_t^a < w_t^s\bar{z}$ for the smart types. (Ignore equality for simplicity here). Due to the segmentation, we must have labour supply in each sector of $n_t^a = \frac{1}{2}\bar{z}, n_t^s = \frac{1}{2}\bar{z}$. Since labour markets have to clear, we get from $N_t^i = n_t^i$ for both sectors $i$ that wages must fulfil $p_t^a A_t = w_t^a$ and $A_t = w_t^s$. This in turn leads to incomes $I^a = p_t^a A_t \bar{z}$ for strong agents and $I^s = A_t \bar{z}$ for smart agents. Their demand respectively is for strong types:

$$c_t^{a,a} = \frac{1}{2} A_t \bar{z} + \frac{\bar{a}}{2} \text{ and } c_t^{s,a} = \frac{1}{2} p_t^a A_t \bar{z} - p_t^a \frac{\bar{a}}{2} = \frac{p_t^a}{2} \max\{(A_t \bar{z} - \bar{a}), 0\}$$

The demand for smart types is:

$$c_t^{a,s} = \frac{1}{2} \frac{A_t \bar{z}}{p_t^a} + \frac{\bar{a}}{2} \text{ and } c_t^{s,s} = \frac{1}{2} A_t \bar{z} - p_t^a \frac{\bar{a}}{2} = \frac{1}{2} \max\{(A_t \bar{z} - p_t^a \bar{a}), 0\}$$

Now consider the goods market for agricultural produce. Demand is given by half a measure of $c_t^{a,a}$ and half a measure of $c_t^{a,s}$:

$$\begin{aligned} c_t^a = \frac{1}{2} c_t^{a,a} + \frac{1}{2} c_t^{a,s} &= \frac{1}{4}(A_t \bar{z} + \bar{a}) + \frac{1}{4}(\frac{A_t \bar{z}}{p_t^a} + \bar{a}) \\ &= \frac{1}{4} A_t \bar{z}\big(1 + \frac{1}{p_t^a}\big) + \frac{1}{2}\bar{a} \end{aligned}$$

That implies a demand for farm labour of $N_t^a = \frac{c_t^a}{A_t} = \frac{1}{4}\bar{z}\big(1 + \frac{1}{p_t^a}\big) + \frac{1}{2}\frac{\bar{a}}{A_t}$. To clear the labour market for farmhands, which sees a supply of $\frac{1}{2}\bar{z}$ in the separating equilibrium, we must have

$$\begin{aligned} \frac{1}{4} A_t \bar{z}\big(1 + \frac{1}{p_t^a}\big) + \frac{1}{2}\frac{\bar{a}}{A_t} &\overset{!}{=} \frac{1}{2}\bar{z} \\ \implies \frac{1}{4}\bar{z}(p_t^a + 1) + \frac{1}{2}\frac{\bar{a}}{A_t} p_t^a &= \frac{1}{2}\bar{z} p_t^a \\ \implies p_t^a &= \frac{\bar{z}}{\bar{z} - 2\frac{\bar{a}}{A_t}} \end{aligned}$$

We can rule out case (iv), because from the indifference condition for strong types and the firm condition we obtain $I_t^a = w_t^s \overset{!}{=} w_t^a \bar{z} = p_t^a A_t \bar{z} \implies p_t^a = \frac{1}{\bar{z}} < 1$. But then from individual demand we have that for any income $I_t$:

$$\begin{aligned} c_t^a &= \frac{1}{2}\frac{I_t}{p_t^a} + \bar{a} > \frac{1}{2}(I_t + \bar{a}) \\ c_t^s &= \max\{\frac{1}{2} I_t - p_t^a \frac{\bar{a}}{2}, 0\} < \frac{1}{2} I_t \end{aligned}$$

So aggregate demand for agricultural goods will be strictly higher than that for services, which is something output under this sorting would fail to accomodate.

We can rule out case (v) where all types work in services, because we have established that $c_t^s > 0 \implies c_t^a > 0$, which would be violated by this sorting.

We can also rule out case (vi), which is some sort of "degenerate sorting". Arithmetic tells us that we can't simultaneously have $I_t^a = w_t^s > w_t^a \bar{z}$ and $I_t^s = w_t^a > w_t^s \bar{z}$ for $\bar{z} > 1$.

(d) **Describe how $p_{a,t}$ and the sorting of workers by sector evolves over time. What happens to the ratio of income for strong types to smart types? Explain your answer.**

If $A_t$ is low, we start out in case (i) or (ii), where all types work in agriculture, with possibly some smart ones working in services. In this case, incomes would be $I_t^a = p_t^a A_t \bar{z}$ and $I_t^s = p_t^a A_t$, so

$$\frac{I_t^a}{I_t^s} = \bar{z}$$

As $A_t$ increases, more and more services are demanded, until we at some point switch to a separating equilibrium as in case (iii). We can read off the cutoff for $A_t$ from prices: In case (ii), $p_t = \bar{z}$, in case (iii), $p_t = \frac{\bar{z}}{\bar{z}-2\frac{\bar{a}}{A_t}}$. So the cutoff for the switch is when $A_t$ grows beyond $A_t^* = 2\frac{\bar{a}}{\bar{z}}$. Then, $I_t^a = p_t^a A_t \bar{z}$ and $I_t^s = A_t \bar{z}$, and

$$\frac{I_t^a}{I_t^s} = p_t^a = \frac{\bar{z}}{\bar{z} - 2\frac{\bar{a}}{A_t}}$$

In the limit, the relative price of agriculture $\lim_{A_t \to \infty} p_t^a = 1$, and so does the ratio of incomes.

In real life, we see that wages in the agricultural sector are typically lower than in services, rather than the other way around. In this model, we held the supply of each factor fixed to be one half each, making smart types a very abundant and hence cheap resource early on in development. However, there is often low supply of skilled labour for developing countries. An interesting case for the purpose of this model could be countries in the former Soviet block, where there tends to be an oversupply of college graduates compared to the availability high-skilled jobs.