ECON 2010c
Lagakos
Fall 2024

Homework 3

Due Date: Wednesday, November 27th, 23:59 hrs, via canvas

Question 1: Skill-Bias in Production

A representative household has preferences given by

$$\sum_{t=0}^{\infty} \beta^t \log(C_t),$$

where $0 < \beta < 1$ and C_t is consumption. The aggregate production function is given by:

$$Y_t = (K_t)^{\alpha}(N_t)^{1-\alpha}$$

where Y_t is output, K_t is capital input, N_t is aggregate labor input and α is capital's share in production. Workers are either skilled workers, N_s, or unskilled workers, N_u, where $N_s + N_u = 1$. The measure of workers that are skilled is exogenous and time invariant. The aggregate labor input is defined as:

$$N_t = \left[(A_{s,t}N_{s,t})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{u,t}N_{u,t})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where ε represents the elasticity of substitution between skilled and unskilled labor inputs, and $A_{s,t}$ and $A_{u,t}$ reflect the time-varying efficiency levels of skilled and unskilled workers. The initial efficiency levels, $A_{s,0}$ and $A_{u,0}$, are given. The laws of motion for the efficiency terms are $A_{s,t+1} = A_{s,t}(1+\gamma_s)$ and $A_{u,t+1} = A_{u,t}(1+\gamma_u)$ where γ_s and γ_u are the growth rates of unskilled and skilled efficiency. The household budget constraint is given by

$$w_{s,t}N_{s,t} + w_{u,t}N_{u,t} + (1-\delta)K_t + r_tK_t = K_{t+1} + C_t$$

where $w_{s,t}$ and $w_{u,t}$ are the competitive wage rates for skilled and unskilled labor and $0 < \delta < 1$.

(a) Define a sequence of markets equilibrium for this economy.

(b) Solve for the skill premium, $w_{s,t}/w_{u,t}$, in the competitive equilibrium. Explain how the long-run behavior of the skill premium depends on the elasticity of substitution, ε.

(c) Suppose that $\gamma_u = \gamma_s$. Explain how the long-run growth rate of Y_t depends on γ_u, ε and α.

Question 2: Endogenous Growth and Human Capital Externalities

An economy is populated by measure one of households whose preferences are:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where β is the discount factor and c_t is consumption at time t. Each household possesses human capital h_t which they supply to the market at price w_t. The production function for c_t is

$$c_t = \phi_t h_t E_t$$

where ϕ_t is the (endogenous) fraction of the household's human capital supplied to the market and E_t is an externality, described below. Consumption is produced by competitive firms that hire human capital, sell output to the households, and take E_t as given each period. The consumption good cannot be stored, and there is no physical capital in the economy. The externality is given by: $E_t = H_t^{\eta}$, where H_t is the average human capital in the economy at t, and η is a positive number. Intuitively, the externality captures the idea that production is more efficient when the average worker is more knowledgable. Households can also supply their labor to the education sector, which is also competitive, and which has production function

$$x_t = A(1-\phi_t)h_t$$

where x_t is the output of new human capital and A is a positive constant. The initial stock of human capital is $h_0 > 0$. The law of motion for human capital is given by $h_{t+1} = h_t + x_t$.

(a) Formulate the household's problem as a dynamic programming problem. Define a recursive competitive equilibrium.

(b) Characterize the balanced growth path of the economy. Characterize the growth rates of consumption and human capital on the balanced growth path.

(c) Now imagine resources are allocated by a benevolent social planner who internalizes the externality. Characterize the balanced growth path under the planner's solution, including the growth rates of c_t and h_t. How do the social planner's allocation and market allocation differ?

Question 3: Planner's Problem in the Romer Model

Consider the following version of the Romer (1990) model. Household preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t),$$

where $0 < \beta < 1$. Consumption and intermediate goods are produced using the following final-goods production function:

$$Y_t = (\phi_t H)^{1-\alpha} \int_0^{A_t} (z_t(i))^{\alpha} di,$$

where $0 < \phi_t < 1$ represents the fraction of human capital devoted to producing output, H is the stock of human capital, A_t is the total stock of varieties in existence, $z_t(i)$ is the quantity of variety i used as an input, and α is a constant satisfying $0 < \alpha < 1$. Each intermediate good in existence can be produced by taking transforming one unit of the consumption good into one unit of the intermediate. New intermediates are created according to:

$$A_{t+1} = A_t + \eta A_t^{\gamma}(1 - \phi_t)H,$$

where η is a positive parameter representing research efficiency, and γ is a parameter governing the extent to which new intermediates get harder and harder to create, satisfying $0 < \gamma \leq 1$. For parts (a)-(c), assume $\gamma = 1$ as in lecture.

(a) Express the social planner's problem as a sequence problem. Note that, unlike in the competitive case, the planner chooses ϕ_t directly.

(b) Solve for ϕ along the balanced growth path. How does ϕ compare to the corresponding competitive-markets allocation of human capital, which is ϕ_{CE} (which we solved for in class)? Briefly describe the intuition for your answer.

(c) Solve for the growth rate of A, $g_A = \frac{A_{t+1}}{A_t}$, along the balanced growth path.

(d) Assume that $\gamma = 0$. Along the balanced growth path, what is the equilibrium increase in varieties each period, i.e. $A_{t+1} - A_t$? What is the equilibrium growth rate of A in this case? Briefly explain the intuition for your answer.

Question 4: Overlapping Generations Model with Social Security

Wait with this question until after the OLG-lecture on Nov 21st.

Consider the following overlapping generations model. The economy is populated by measure one of households that live for two periods. Households are endowed with one unit of labor when young, which they supply inelastically, and no time endowment when old. The preferences of an agent born at time t are:

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t),$$

where c_t^t and c_{t+1}^t are consumption when young and when old, and $0 < \beta < 1$. Households save capital k_{t+1}^t when young and earn rental income from that capital when old. The rental rate at t is denoted r_t, and the wage rate is w_t. The production technology is $Y_t = AK_t^\theta N_t^{1-\theta}$, where K_t and N_t are aggregate capital and labor inputs, $0 < \theta < 1$, and A is a positive constant. Markets are perfectly competitive, and the production technology is operated by competitive firms. Capital depreciates fully each period.

The government runs a "pay as you go" social security system that takes d_t units of the final good from the young in period t and redistributes it to the old in period t in the form of a benefit, b_t, where $d_t = b_t$. Assume that the entire sequence of transfers, $\{d_t\}_{t=0}^\infty$, is known to all agents and feasible (meaning the transfer is less than wage income for the young in each period). There is an initial old generation that is endowed with capital k_0^{-1}.

(a) Write the household's problem and firm's problem and characterize the optimality conditions for the household and firm. Define a sequence-of-markets equilibrium.

(b) Suppose the social-security benefit is a constant value, $\bar{b}$, each period, equal to a fraction γ times the steady-state capital stock, where $0 < \gamma < 1$. Solve for the steady-state capital stock, consumption of the young, and consumption of the old.

(c) Suppose $\beta = 0.9$, $\gamma = 0.1$, $A = 1$, $\theta = 0.3$. Does the presence of the "pay as you go" social security system increase consumption relative to the steady state without social security? Provide a brief intuition for your results. You may want to draw on the solution to a planner's problem.