ECON 2010C — Problem Set 3 – Suggested Solutions

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Question 1: Skill-Bias in Production

A representative household has preferences given by

$$\sum_{t=0}^{\infty} \beta^t \log(C_t),$$

where $0 < \beta < 1$ and C_t is consumption. The aggregate production function is given by:

$$Y_t = (K_t)^{\alpha}(N_t)^{1-\alpha}$$

where Y_t is output, K_t is capital input, N_t is aggregate labor input and α is capital's share in production. Workers are either skilled workers, N_s, or unskilled workers, N_u, where $N_s + N_u = 1$. The measure of workers that are skilled is exogenous and time invariant. The aggregate labor input is defined as:

$$N_t = \left[(A_{s,t}N_{s,t})^{\frac{\epsilon-1}{\epsilon}} + (A_{u,t}N_{u,t})^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}},$$

where ϵ represents the elasticity of substitution between skilled and unskilled labor inputs, and $A_{s,t}$ and $A_{u,t}$ reflect the time-varying efficiency levels of skilled and unskilled workers. The initial efficiency levels, $A_{s,0}$ and $A_{u,0}$, are given. The laws of motion for the efficiency terms are $A_{s,t+1} = A_{s,t}(1 + \gamma_s)$ and $A_{u,t+1} = A_{u,t}(1 + \gamma_u)$ where γ_s and γ_u are the growth rates of unskilled and skilled efficiency. The household budget constraint is given by

$$w_{s,t}N_{s,t} + w_{u,t}N_{u,t} + (1 - \delta)K_t + r_tK_t = K_{t+1} + C_t$$

where $w_{s,t}$ and $w_{u,t}$ are the competitive wage rates for skilled and unskilled labor and $0 < \delta < 1$.

(a) **Define a sequence of markets equilibrium for this economy.**

The household faces the following problem, taking the sequence of wages $\{w_{u,t}, w_{s,t}\}_{t=0}^{\infty}$ and rental rates $\{r_t\}_{t=0}^{\infty}$ as given:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad s.t.\ \forall t$$
$$c_t + k_{t+1} = w_{s,t}N_{s,t} + w_{u,t}N_{u,t} + (1-\delta)k_t + r_t k_t$$

The firm problem is, taking wages and rental rates as given:

$$\max_{N^d_{s,t}, N^d_{u,t}, K_t} K_t^{\alpha}\Big(\Big[(A_{s,t}N_{s,t})^{\frac{\varepsilon-1}{\varepsilon}} + (A_{u,t}N_{u,t})^{\frac{\varepsilon-1}{\varepsilon}}\Big]^{\frac{\varepsilon}{\varepsilon-1}}\Big)^{1-\alpha} - w_{s,t}N^d_{s,t} - w_{u,t}N^d_{u,t} - r_t K_t$$

The competitive equilibrium consists of

(1) Household choices $\{c_t, k_{t+1}\}_{t=0}^{\infty}$
(2) Firm factor demand $\{N_{u,t}, N_{s,t}\}_{t=0}^{\infty}$
(3) Prices and rental rates $\{w_{u,t}, w_{s,t}, r_t\}_{t=0}^{\infty}$

such that

(i) Given (3), households solve their household problem, yieling (1)
(ii) Given (3), firms solve their profit problem, yielding (2)
(iii) Markets clear $\forall t$ (one of the conditions is algebraically redundant):
  - Goods market: $c_t + k_{t+1} = f(k_t, N_{u,t}, N_{s,t}) + (1-\delta)k_t$
  - Factor market for capital: $K_t = k_t$
  - Factor market for unskilled labour: $N^d_{u,t} = N_{u,t}$
  - Factor market for skilled labour: $N^d_{s,t} = N_{s,t}$

(b) **Solve for the skill premium, $w_{s,t}/w_{u,t}$, in the competitive equilibrium. Explain how the long-run behavior of the skill premium depends on the elasticity of substitution, ϵ.**

For this, it suffices to look at the firm f.o.c.s w.r.t. labour only:

$$[N^d_{u,t}]\ (1-\alpha)\Big(\frac{K_t}{N_t}\Big)^{\alpha} \frac{\varepsilon}{\varepsilon-1} N_t^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} N_{u,t}^{\frac{\varepsilon-1}{\varepsilon}-1} A_{u,t}^{\frac{\varepsilon-1}{\varepsilon}} = w_{u,t}$$
$$[N^d_{s,t}]\ (1-\alpha)\Big(\frac{K_t}{N_t}\Big)^{\alpha} \frac{\varepsilon}{\varepsilon-1} N_t^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} N_{s,t}^{\frac{\varepsilon-1}{\varepsilon}-1} A_{s,t}^{\frac{\varepsilon-1}{\varepsilon}} = w_{s,t}$$

Now taking the ratio will yield the ratio $\frac{w_{s,t}}{w_{u,t}}$ need, and also get rid of a lot of common things, since this is about allocation within labour N_t, not versus capital:

$$\begin{aligned}
\frac{w_{s,t}}{w_{u,t}} &= \frac{N_{s,t}^{\frac{\varepsilon-1}{\varepsilon}-1} A_{s,t}^{\frac{\varepsilon-1}{\varepsilon}}}{N_{u,t}^{\frac{\varepsilon-1}{\varepsilon}-1} A_{u,t}^{\frac{\varepsilon-1}{\varepsilon}}} \\
&\propto \left(\frac{A_{s,t}}{A_{u,t}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \\
&\propto \left(\frac{A_{s,0}}{A_{u,0}}\right)^{\frac{\varepsilon-1}{\varepsilon}} \left(\frac{1+\gamma_s}{1+\gamma_u}\right)^{t\cdot\frac{\varepsilon-1}{\varepsilon}}
\end{aligned}$$

where the fixed factor of proportionality is given by the relative abundance of labour types. In the long run, i.e. $t \to \infty$, the behaviour of the skill-premium depends on the growth rates γ_u vs. γ_s and ε.

Let us assume that $\gamma_s > \gamma_u$ (the reverse case is opposite): Then, if $\frac{\varepsilon-1}{\varepsilon} \geq 0$, that is, if $\varepsilon \geq 1$ or $\varepsilon < 0$, then the skill premium will blow up to ∞:

$$\lim_{t\to\infty} \frac{w_{s,t}}{w_{u,t}} \propto \lim_{t\to\infty} \left(\frac{1+\gamma_s}{1+\gamma_u}\right)^{t\cdot\frac{\varepsilon-1}{\varepsilon}} = \infty$$

The intuition is that under gross substitutes, since skilled labour becomes more and more productive in relative terms, more value added is born by the latter, and hence the owners of that factor are better and better off than the owner of the less productive factor.

Now consider when $\frac{\varepsilon-1}{\varepsilon} < 1$, i.e. $\varepsilon \in (0,1)$, or: under gross complementarity:

$$\lim_{t\to\infty} \frac{w_{s,t}}{w_{u,t}} \propto \lim_{t\to\infty} \left(\frac{1+\gamma_s}{1+\gamma_u}\right)^{t\cdot\frac{\varepsilon-1}{\varepsilon}} = 0$$

Here, the skilled factor becomes more and more abundant, but due to complementarity we would like unskilled labour to grow at the same rate. Since it does not, in equilibrium the price for unskilled labour goes through the roof, thus causing the wage premium to go to zero.

In the limit case where $\varepsilon \to 1$, that gives us Cobb-Douglas, and the wage premium will equal the factor of proportionality:

$$\lim_{t\to\infty} \frac{w_{s,t}}{w_{u,t}} = \frac{N_{s,t}^{\frac{\varepsilon-1}{\varepsilon}-1}}{N_{u,t}^{\frac{\varepsilon-1}{\varepsilon}-1}} \left(\frac{A_{s,0}}{A_{u,0}}\right)^{\frac{\varepsilon-1}{\varepsilon}}$$

The intuition is the usual one: With Cobb-Douglas, each factor sees constant (expenditure) shares.

(c) **Suppose that $\gamma_u = \gamma_s$. Explain how the long-run growth rate of Y_t depends on γ_u, ϵ and α.**

To get the long-run growth rate, it is easiest to just solve the planner's problem and then characterise the BGP. I will use γ_u to denote $(1+\gamma_u)$ as in the question, to save us some typing.

$$V(A,k) = \max_{c,k'} \; \log(c) + \beta V(A',k') \quad s.t. \; c + k' = k^\alpha N^{1-\alpha} + (1-\delta)k$$

Where $A_t := \gamma_u^t$. First, we note that given our CES labour aggregator and under the assumption that $\gamma_u = \gamma_s$, that $N_t = \gamma_u^t N_0$, and we also have $N = AN_0$, or: $N' = \gamma_u N$. Lagrangian:

$$\begin{aligned}
V(A,k) =& \max_{c,k'} \; \log(c) + \beta V(A',k') + \lambda\big[k^\alpha (AN_0)^{1-\alpha} + (1-\delta)k - c - k'\big] \\
[c] :& \; \frac{1}{c} = \lambda \\
[k'] :& \; \beta \frac{\partial}{\partial k'} V(A',k') = \lambda \\
[EnvThm] :& \; \frac{\partial}{\partial k} V(A,k) = \lambda\big[\alpha k^{\alpha-1}(AN_0)^{1-\alpha} + (1-\delta)\big] \\
&\longrightarrow \frac{\partial}{\partial k'} V(A',k') = \lambda'\big[\alpha (k')^{\alpha-1}(A'N_0)^{1-\alpha} + (1-\delta)\big] \\
[\lambda] :& \; k^\alpha (AN_0)^{1-\alpha} + (1-\delta)k = c + k'
\end{aligned}$$

Reducing that to the usual equations:

$$\frac{1}{c} = \beta \frac{1}{c'}\big[\alpha (k')^{\alpha-1}(A'N_0)^{1-\alpha} + (1-\delta)k\big] \qquad \text{(Euler equation)}$$

$$k^\alpha (AN_0)^{1-\alpha} + (1-\delta)k = c + k' \qquad \text{(resource constraint)}$$

Imposing a BGP, starting with the Euler equation:

$$\gamma_c = \frac{c'}{c} = \beta\Big[\alpha\Big(\frac{k'}{A'N_0}\Big)^{\alpha-1} + (1-\delta)\Big]$$

Which yields that $\gamma_k = \gamma_u$ and $\gamma_c = \beta\Big[\alpha\Big(\frac{k}{AN_0}\Big)^{\alpha-1} + (1-\delta)k\Big]$. From the resource constraint:

$$\begin{aligned}
& k^\alpha (AN_0)^{1-\alpha} + (1-\delta)k = c + k' \\
\Longrightarrow & \Big(\frac{k}{AN_0}\Big)^{\alpha-1} + (1-\delta) = \frac{c}{k} + \gamma_k
\end{aligned}$$

We already know that the LHS fraction is a constant on the BGP, so $\gamma_c = \gamma_k$. Since γ_u is given, we have fully characterised the BGP.

Question 2: Endogenous Growth and Human Capital Externalities

An economy is populated by measure one of households whose preferences are:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where β is the discount factor and c_t is consumption at time t. Each household possesses human capital h_t which they supply to the market at price w_t. The production function for c_t is

$$c_t = \phi_t h_t E_t$$

where ϕ_t is the (endogenous) fraction of the household's human capital supplied to the market and E_t is an externality, described below. Consumption is produced by competitive firms that hire human capital, sell output to the households, and take E_t as given each period. The consumption good cannot be stored, and there is no physical capital in the economy. The externality is given by: $E_t = H_t^\eta$, where H_t is the average human capital in the economy at t, and η is a positive number. Intuitively, the externality captures the idea that production is more efficient when the average worker is more knowledgable. Households can also supply their labor to the education sector, which is also competitive, and which has production function

$$x_t = A(1-\phi_t)h_t$$

where x_t is the output of new human capital and A is a positive constant. The initial stock of human capital is $h_0 > 0$. The law of motion for human capital is given by $h_{t+1} = h_t + x_t$.

(a) **Formulate the household's problem as a dynamic programming problem. Define a recursive competitive equilibrium.**

The household faces the following recursive problem with the state variables h for individual human capital and H for aggregate human capital: Given wages w and the relative price for education p, the household optimises:

$$\begin{aligned} V(h,H) = \max_{c,x,h'} \ & \log(c) + \beta V(h',H') \quad s.t. \\ & c + p(H)x = w(H)h \\ & h' = h + x \\ & H' = \hat{G}(H) \end{aligned}$$

where $\hat{G}$ is some belief about the law of motion for H. Note that households rent out their entire human capital $h = \phi h + (1-\phi)h$ to a common labour market. We have discussed in a previous problem set how this is more straightforward than having separated labour markets and using equilibrium conditions to equalize wages.

Production firms maximise their static profit objective:

$$\max_{\phi h} \quad \phi h H^{\eta} - w(H)\phi h$$

The firm just chooses the amount of human capital $N_1^d = \phi h$ it demands.

Harvard Corporationy also maximises its profit objective:

$$\max_{(1-\phi)h} p(H)A(1-\phi)h - w(H)(1-\phi)h$$

The university just chooses the amount of education workers (PhD students and Davids) $N_2^d = (1-\phi)h$ it demands.

The recursive competitive equilibrium consists of

(1) Household value function $V(h,H)$ and policy functions $c(h,H), x(h,H), h'(h,H)$
(2) Production firm factor demand $N_1^d = \phi H$
(3) Education firm factor demand $N_2^d = (1-\phi)H$
(4) Price functions $p(H), w(H)$
(5) A belief on the law of motion for aggregate human capital $H' = \hat{G}(H)$

such that

(a) Given (4) and (5), households solve their household problem, yielding (1).
(b) Given (4), production firms solve their profit maximisation problem, yielding (2)
(c) Given (4), educaton firms solve their profit maximisation problem, yielding (3)
(d) Markets clear (one condition is algebraically redundant):
    - Consumption goods: $c(H,H) = \underbrace{N_1^d}_{\phi H} H^{\eta}$
    - Education goods: $x(H,H) = A \underbrace{N_2^d}_{(1-\phi)H}$
    - Factor market (labour): $N_1^d + N_2^d = H$
(e) Beliefs are correct in equilibrium: $h'(H,H) = \hat{G}(H)$

(b) **Characterize the balanced growth path of the economy. Characterize the growth rates of consumption and human capital on the balanced growth path.**

Recall the household problem:

$$
\begin{aligned}
V(h,H) = \max_{c,x,h'} \; & \log(c) + \beta V(h',H') \quad s.t. \\
& c + p(H)x = w(H)h \\
& h' = h + x \\
& H' = \hat{G}(H)
\end{aligned}
$$

Lagrangian for the household, where we substitute out $x = h' - h$:

$$
\begin{aligned}
\mathcal{L} =& \log(c) + \beta V(h',H') + \lambda[w(H)h - c - p(H)(h'-h)] \\
[c] :& \frac{1}{c} = \lambda \\
[h'] :& \beta \frac{\partial}{\partial h'} V(h',H') = \lambda p(H) \\
[EnvThm] :& \frac{\partial}{\partial h} V(h,H) = \lambda(w(H) + p(H)) \longrightarrow \frac{\partial}{\partial h'} V(h',H') = \lambda'(w(H') + p(H')) \\
[\lambda] :& w(H)h = c + p(H)x
\end{aligned}
$$

This gives us

$$
\frac{1}{c} p(H) = \beta \frac{1}{c'}(w(H) + p(H)) \qquad \text{(Euler Equation)}
$$

$$
w(H)h = c + p(H)(h' - h) \qquad \text{(budget constraint)}
$$

If the household invests a marginal unit less into education today, it receives p and can spend it on consumption, yielding a marginal change in utility of $\frac{1}{c}$. At the optimum, that is balanced against the consideration of spending one dollar more on education, where the benefit is more human capital tomorrow, leading to higher income through additional wage $w(H')$ and by a potentially higher value of human capital, all converted to "utils" by multiplying with future marginal utility. The notion of human "capital" can be seen easier when rewritten to:

$$
\frac{c'}{c} = \beta \frac{w(H') + p(H')}{p(H)}
$$

where we have the familiar Euler equation, but instead of the return to physical capital we now have $\frac{w(H')+p(H')}{p(H)}$, which very much resembles the classic decomposition of asset returns into of dividends (wages) and price appreciation.

From the firm problems we get that $w(H) = H^\eta$ and $w(H) = p(H)A \Longrightarrow p(H) = \frac{w(H)}{A}$, assuming an interior solution where $x > 0$, $\phi \in (0,1)$. This pins down the wage and price functions. We further have market clearing conditions:

$$
\begin{aligned}
c &= \phi h H^\eta \\
h' - h &= A(1-\phi)h
\end{aligned}
$$

To characterise BGP growth rates, conjecture that $\gamma_\phi = 1$, i.e. $\phi_t = \phi$ constant:
In the Euler equation:

$$\begin{aligned}
\frac{c'}{c} &= \beta\Big(\frac{w(H') + p(H')}{p(H)}\Big) \\
\implies \gamma_c &= \beta\Big(\frac{(H')^\eta + \frac{(H')^\eta}{A}}{\frac{H^\eta}{A}}\Big) \\
\implies \gamma_c &= \beta\Big(\frac{(H')^\eta A + (H')^\eta}{H^\eta}\Big) \\
\implies \gamma_c &= \beta\big(A+1\big)(\gamma_H)^\eta
\end{aligned}$$

Now in equilibrium $h = H$, and we can use the market clearing condition:

$$\begin{aligned}
c &= \phi H^{\eta+1} \\
\implies \quad \frac{c'}{c} &= \Big(\frac{H'}{H}\Big)^{\eta+1} \\
\implies \quad \gamma_c &= \gamma_H^{\eta+1}
\end{aligned}$$

These are two equations in two unknowns, and we solve for:

$$\begin{aligned}
\gamma_c = (\gamma_H)^{\eta+1} &\overset{!}{=} \beta(A+1)(\gamma_H)^\eta \\
\implies \gamma_H &= \beta(A+1) \\
\gamma_c &= (\beta(A+1))^{\eta+1}
\end{aligned}$$

To obtain the BGP dedication to production ϕ^{CE}:

$$\begin{aligned}
h' - h = A(1-\phi)h \implies \frac{h'-h}{h} &= \gamma_H - 1 = A(1-\phi) \\
&\implies \beta(A+1) - 1 = A(1-\phi) \\
&\implies \phi = 1 - \frac{\beta(A+1)-1}{A} \\
&\implies \phi^{CE} = \frac{A - \beta - \beta A + 1}{A}
\end{aligned}$$

(c) **Now imagine resources are allocated by a benevolent social planner who internalizes the externality. Characterize the balanced growth path under the planner's solution, including the growth rates of c_t and h_t. How do the social planner's allocation and market allocation differ?**

The planner's problem is:

$$\begin{aligned}
V(H) = \max_{c,\phi,H'} \log(c) + \beta V(H') \quad s.t.& \\
c &= \phi H^{\eta+1} \\
x &= A(1-\phi)H \\
H' &= x + H
\end{aligned}$$

The Lagrangian:

$$\begin{aligned}
\mathcal{L} = \log(c) + \beta V(H') + \mu\big[H + A(1-\phi)H - H'\big] + \lambda\big[\phi H^{\eta+1} - c\big]& \\
[c] : \frac{1}{c} &= \lambda \\
[H' :]\ \beta \frac{\partial}{\partial H'} V(H') &= \mu \\
[\phi] : \mu A H = \lambda H^{\eta+1} &\implies \mu = \frac{\lambda H^{\eta}}{A} \\
[EnvThm] : \frac{\partial}{\partial H} V(H) &= \mu\big[1 + A(1-\phi)\big] + \lambda\phi(\eta+1)H^{\eta} \\
&= \frac{\lambda H^{\eta}}{A}\big[1 + A(1-\phi)\big] + \lambda\phi(\eta+1)H^{\eta} \\
&= \lambda H^{\eta}\big[\frac{1}{A} + 1 - \phi + \phi(\eta+1)\big] \\
&= \lambda H^{\eta}\big[\frac{1}{A} + 1 + \phi\eta\big] \\
\longrightarrow \frac{\partial}{\partial H'} V(H') &= \lambda'(H')^{\eta}\big[\frac{1}{A} + 1 + \phi\eta\big] \\
[\mu] &: \ \dots \\
[\lambda] &: \ \dots
\end{aligned}$$

Yielding

$$\begin{aligned}
[EE] : \beta\Big(\frac{H'}{H}\Big)^{\eta}\big[1 + A + A\phi\eta\big] &= \frac{c'}{c} \\
[\mu] : \frac{H'}{H} &= 1 + A(1-\phi) \\
[\lambda] : c &= \phi H^{\eta+1}
\end{aligned}$$

Imposing balanced growth:

$$\begin{aligned}
[EE] : \beta(\gamma_H)^{\eta}\big[1 + A + A\phi\eta\big] &= \gamma_c \\
[\mu] : \gamma_c &= 1 + A(1-\phi) \\
[\lambda] : c = \phi H^{\eta+1} &\implies \gamma_c = (\gamma_H)^{\eta+1}
\end{aligned}$$

Equating $[EE]$ and $[\lambda]$:

$$\begin{aligned}\beta(\gamma_H)^\eta[1+A+A\phi\eta] &= (\gamma_H)^{\eta+1}\\ \implies \beta[1+A+A\phi\eta] &= \gamma_H\end{aligned}$$

Combine with $[\mu]$:

$$\begin{aligned}\beta[1+A+A\phi\eta] = \gamma_H &= 1+A(1-\phi)\\ \implies \beta[1+A]-(1+A) &= \phi(-A-\beta A\eta)\\ \implies \phi &= \frac{\beta(1+A)-(1+A)}{-A-\beta A\eta}\\ &= \frac{A+1-\beta-\beta A}{A+\beta A\eta}\end{aligned}$$

Compared to $\phi^{CE} = \frac{A+1-\beta-\beta A}{A}$, the denominator of the planner is higher, which means that the planner puts more emphasis on innovation rather than production. This is consistent with the intuition that the decentralised agents fail to internalise that higher human capital is good for future productivity, and accordingly underinvest to it relative to first-best.
We can also see that if there was no externality ($\eta = 0$), we return to first-best: $\phi^{SP}_{\eta=0} = \frac{A+1-\beta-\beta A}{A} = \phi^{CE}$.

Question 3: Planner's Problem in the Romer Model

Consider the following version of the Romer (1990) model. Household preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t),$$

where $0 < \beta < 1$. Consumption and intermediate goods are produced using the following final-goods production function:

$$Y_t = (\phi_t H)^{1-\alpha} \int_0^{A_t} (z_t(i))^\alpha di,$$

where $0 < \phi_t < 1$ represents the fraction of human capital devoted to producing output, H is the stock of human capital, A_t is the total stock of varieties in existence, $z_t(i)$ is the quantity of variety i used as an input, and α is a constant satisfying $0 < \alpha < 1$. Each intermediate good in existence can be produced by taking transforming one unit of the consumption good into one unit of the intermediate. New intermediates are created according to:

$$A_{t+1} = A_t + \eta A_t^\gamma (1-\phi_t)H,$$

where η is a positive parameter representing research efficiency, and γ is a parameter governing the extent to which new intermediates get harder and harder to create, satisfying $0 < \gamma \leq 1$. For parts (a)-(c), assume $\gamma = 1$ as in lecture.

(a) **Express the social planner's problem as a sequence problem. Note that, unlike in the competitive case, the planner chooses ϕ_t directly.**

Note: We make the timing assumption on intermediate goods that we need them on-hand at the start of the period to produce anything. An alternative would be a roundabout production structure, where the output in a period simultaneously is used as input, producing output in the first place. This is usually well-defined as long as some share is consumed. Here, we'll stick to the first assumption, as this is what David did during lecture. That makes intermediate goods behave a bit like capital we have to save for next period, except that it fully depreciates during production.

The planner solves:

$$\max_{\{c_t,\phi_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad s.t.$$

$$\begin{aligned} &Y_t = (\phi_t H)^{1-\alpha} \int_0^{A_t} (z_i(t))^{\alpha}\, di \\ &\int_0^{A_t} z_t(i) di = k_t \\ &A_{t+1} = A_t + \eta A_t^{\gamma}(1-\phi_t)H \\ &c_t + k_{t+1} = Y_t \end{aligned}$$

(b) **Solve for ϕ along the balanced growth path. How does ϕ compare to the corresponding competitive-markets allocation of human capital, which is ϕ_{CE} (which we solved for in class)? Briefly describe the intuition for your answer.**

Before we proceed, we can already take of static optimality in intermediate good usage. We can separate it from the rest of the problem because $z_t(i)$ is a purely static choice that can be solved independently of our choices over ϕ and so on. Given some saved up intermediate good k_t, how should we distribute the inputs across our A_t varieties to maximise output? Try to think intuitively what the answer should be. Mathematically, we solve:

$$\begin{aligned}\max_{\{z_t(i)\}_{i\in[0,A_t]}} \quad & (\phi_t H)^{1-\alpha}\int_0^{A_t} (z_i(t))^\alpha \, di \quad s.t. \quad \int_0^{A_t} z_t(i)di = k_t \\ \mathcal{L} =& (\phi_t H)^{1-\alpha}\int_0^{A_t} (z_i(t))^\alpha \, di + \lambda\big(k_t - \int_0^{A_t} z_t(i)di\big) \\ [z_t(i)] :& \ (\phi_t H)^{1-\alpha}\alpha z_t(i)^{\alpha-1} = \lambda \quad \forall i \\ [\lambda] :& \ \int_0^{A_t} z_t(i)di = k_t\end{aligned}$$

Dividing the condition for $z_t(i)$ by the one for some $z_t(j)$, we obtain:

$$\Big(\frac{z_t(i)}{z_t(j)}\Big)^{\alpha-1} = 1$$

Since $\alpha \neq 1$, we can infer that $z_t(i) = z_t(j) = z^* \ \forall i, j \in [0, A_t]$. Use $[\lambda]$ to get:

$$\begin{aligned} k_t &\overset{!}{=} \int_0^{A_t} z_t(i)di \\ &= \int_0^{A_t} z^* di \Longrightarrow z^* = \frac{k_t}{A_t}\end{aligned}$$

which is intuitive, and subsequently

$$\begin{aligned} Y_t &= (\phi_t H)^{1-\alpha}\int_0^{A_t} \big(\frac{k_t}{A_t}\big)^\alpha \, di \\ &= (\phi_t H)^{1-\alpha} k_t^\alpha A_t^{1-\alpha}\end{aligned}$$

It is important to understand why a higher level of technology increases production here. The key is that each variety has diminishing marginal products due to the α-exponent. Having an additional variety of intermediates allows for more efficient usage of the stock of intermediates k_t, hence more output Y_t.

I will now solve the overall problem recursively, because I hate typing subscripts. The

Lagrangian is:

$$\begin{aligned}
V(A,k) &= \max_{c,\phi,k'} \log(c) + \beta V(A',k') \quad s.t. \\
&\quad \begin{cases} (\phi H)^{1-\alpha} k^{\alpha} A^{1-\alpha} - c - k' \\ A' = A + \eta A(1-\phi)H \end{cases} \\
\mathcal{L} &= \log(c) + \beta V(A',k') + \lambda\big[(\phi H)^{1-\alpha} k^{\alpha} A^{1-\alpha} - c - k'\big] + \\
&\qquad\qquad\qquad\qquad \mu\big[A + \eta A(1-\phi)H - A'\big] \\
[c] &: \frac{1}{c} = \lambda \\
[\phi] &: \lambda(1-\alpha)\phi^{-\alpha} H^{1-\alpha} k^{\alpha} A^{1-\alpha} = \mu\eta A H \\
[A'] &: \beta \frac{\partial}{\partial A'} V(A',k') = \mu \\
[k'] &: \beta \frac{\partial}{\partial k'} V(A',k') = \lambda \\
[EnvThm] &: \frac{\partial}{\partial A} V(A,k) = \lambda\big[(\phi H)^{1-\alpha}(1-\alpha)A^{-\alpha}k^{\alpha}\big] + \mu\big[1 + \eta(1-\phi)H\big] \\
&\longrightarrow \frac{\partial}{\partial A'} V(A',k') = \lambda'\big[(\phi' H)^{1-\alpha}(1-\alpha)\big(\frac{k'}{A'}\big)^{\alpha}\big] + \mu'\big[1 + \eta(1-\phi')H\big] \\
[EnvThm] &: \frac{\partial}{\partial k'} V(A',k') = \lambda\big[(\phi H)^{1-\alpha}\alpha\big(\frac{k}{A}\big)^{\alpha-1}\big] \\
&\longrightarrow \frac{\partial}{\partial k'} = \lambda'\big[(\phi' H)^{1-\alpha}\alpha\big(\frac{k'}{A'}\big)^{\alpha-1}\big] \\
[\lambda] &: c + k' = (\phi H)^{1-\alpha} k^{\alpha} A^{1-\alpha} \\
[LoM] &: A' = A + \eta A(1-\phi)H
\end{aligned}$$

We can use $[\phi]$ to express the Lagrangian on the law of motion for technology μ in terms of the Lagrangian for production:

$$\begin{aligned}
&[\phi] : \lambda(1-\alpha)\phi^{-\alpha} H^{1-\alpha} k^{\alpha} A^{1-\alpha} = \mu\eta A H \\
&\Longrightarrow \lambda(1-\alpha)\phi^{-\alpha} H^{-\alpha} k^{\alpha} A^{-\alpha} = \mu\eta \\
&\Longrightarrow \mu = \frac{\lambda(1-\alpha)}{\eta(\phi H)^{\alpha}}\Big(\frac{k}{A}\Big)^{\alpha}
\end{aligned}$$

Albeit it looks complicated, the intuition for this expression comes from the $[\phi]$-f.o.c.

It yields for the Envelope condition:

$$\begin{aligned}[EnvThm]:\ \frac{\partial}{\partial A'}V(A',k') &= \lambda'\big[(\phi' H)^{1-\alpha}(1-\alpha)\big(\frac{k'}{A'}\big)^{\alpha}\big] + \mu'\big[1+\eta(1-\phi')H\big] \\ &= \lambda'\big[(\phi' H)^{1-\alpha}(1-\alpha)\big(\frac{k'}{A'}\big)^{\alpha}\big] + \frac{\lambda'(1-\alpha)}{\eta(\phi' H)^{\alpha}}\Big(\frac{k'}{A'}\Big)^{\alpha}\big[1+\eta(1-\phi')H\big] \\ &= \frac{\lambda'(1-\alpha)}{\eta(\phi' H)^{\alpha}}\Big(\frac{k'}{A'}\Big)^{\alpha}\Big[\eta(\phi' H)+1+\eta(1-\phi')H\Big]\end{aligned}$$

Plugging into the Euler equation:

$$\begin{aligned}[EE-A']:\beta\Big[\frac{\lambda'(1-\alpha)}{\eta(\phi' H)^{\alpha}}\Big(\frac{k'}{A'}\Big)^{\alpha}\Big[\eta(\phi' H)+1+\eta(1-\phi')H\Big]\Big] &= \frac{\lambda(1-\alpha)}{\eta(\phi H)^{\alpha}}\Big(\frac{k}{A}\Big)^{\alpha} = \mu \\ \implies \beta\Big[\Big(\frac{k'}{k}\Big)^{\alpha}\Big(\frac{A}{A'}\Big)^{\alpha}\big[1+\eta H\big]\Big] &= \frac{\lambda}{\lambda'}\Big(\frac{\phi'}{\phi}\Big)^{\alpha} \\ [EE-k']:\beta\Big[\lambda'\big[(\phi' H)^{1-\alpha}\alpha\big(\frac{k'}{A'}\big)^{\alpha-1}\big]\Big] &= \lambda \\ \implies \beta\Big[(\phi' H)^{1-\alpha}\alpha\big(\frac{k'}{A'}\big)^{\alpha-1}\Big] &= \frac{\lambda}{\lambda'}\end{aligned}$$

The first Euler equation w.r.t. A' is a bit harder to interpret: It encapsulates the planner's trade-off between putting people to work, yielding utility through consumption today, and putting them to innovate, thus increasing the technological frontier for tomorrow. Most of that intuition is encapsulated in the optimality of ϕ, which we have plugged in here: It compares the marginal value of higher productivity (marginal product of working in production, converted to consumption utils by λ) with the marginal value of having a higher level of technology (marginal product of innovation effort, i.e. increase A' by ηAH, converted into utils by μ). What makes it look a bit awkward is that – physically – putting more labour into innovation only bears fruits for A' tomorrow, not A today. But the ϕ optimality is between λ and μ, not some μ'? Here we note that the choice of A' does affect our value function today, which captures the entire present-discounted utility stream. If I change my choice of A', that changes my value function today, and the Lagrange multiplier μ says by how much. This timing interpretation of μ is made explicit by $[A']$, which says that the marginal change in the continuation value is exactly μ. Ultimately, the choice of A' vs. c and ϕ vs. c are two sides of the same coin.

The second Euler equation w.r.t. k' encapsulates the standard trade-off of consumption today versus production tomorrow. The logic is the same as in the neoclassical growth model. In fact, we can view the savings just like capital before (I chose the notation k_t on purpose), since the timing and function is the same. The only major difference is that this kind of "capital" depreciates fully during usage: $\delta = 1$. The trade-off should look familiar to you: If too much output is consumed, little can be produced tomorrow. In the decentralised equilibrium, households were keen to save due to the monetary return on savings, here it is the marginal product of capital directly.

Now imposing a BGP, where we conjecture that $\phi' = \phi$:

$$\begin{aligned}
[EE - A'] &: \beta\Big(\frac{\gamma_k}{\gamma_A}\Big)^{\alpha}(1+\eta H) = \gamma_c \\
[EE - k'] &: \beta\Big[(\phi' H)^{1-\alpha}\alpha\big(\frac{k'}{A'}\big)^{\alpha-1}\Big] = \gamma_c \\
[RC] &: c + k' = (\phi H)^{1-\alpha}A^{1-\alpha}k^{\alpha} \\
&\implies (\phi' H)^{1-\alpha}\alpha\big(\frac{k'}{A'}\big)^{\alpha-1} = \alpha\frac{c' + k''}{k'} = \alpha(\frac{c'}{k'} + \gamma_k) \\
[LoM] &: A' = A + \eta A(1-\phi)H \\
&\implies \gamma_A = \frac{A'}{A} = 1 + \eta(1-\phi)H
\end{aligned}$$

We can infer from $[EE - k']$ combined with $[RC]$ that $\beta\alpha\big(\frac{c'}{k'} + \gamma_k\big) = \gamma_c$, which implies that $\gamma_c = \gamma_k$. Using that in $[EE - A']$, we get that $\gamma_c = \beta(1+\eta H)$. From $[EE - k']$, we further get that $\gamma_k = \gamma_A$, so that in particular $\gamma_c = \gamma_k = \gamma_A$. As a last step:

$$\begin{aligned}
&\beta(1+\eta H) = \gamma_c = \gamma_A = 1 + \eta(1-\phi)H \\
\implies &\eta H\phi = 1 - \beta + \eta H - \beta\eta H = (1-\beta)(1+\eta H) \\
\implies &\phi^{SP} = \frac{(1-\beta)(1+\eta H)}{\eta H}
\end{aligned}$$

The growth rate of technology is, accordingly:

$$\begin{aligned}
\gamma_A &= 1 + \eta(1-\phi^{SP})H \\
&= \ldots = \beta + \eta\beta H
\end{aligned}$$

Let us contrast that with the allocation coming out of the competitive equilibrium:

$$\begin{aligned}
\phi^{CE} &= \frac{(1-\beta) + \eta H}{\eta H(\alpha\beta + 1)} \\
\phi^{SP} &= \frac{(1-\beta)(1+\eta H)}{\eta H}
\end{aligned}$$

On first glance, we may not be able to order these two. However:

$$\begin{aligned}
&\phi^{SP} = (\alpha\beta + 1)\phi^{CE} - \beta \\
\implies &\phi^{CE} - \phi^{SP} = \beta - \alpha\beta\phi^{CE} = \beta(1 - \alpha\phi^{CE})
\end{aligned}$$

Since $\alpha \in (0,1)$ and $\phi^{CE} \in [0,1]$, we have that $\phi^{CE} > \phi^{SE}$. That is, on the balanced growth path, the decentralised equilibrium allocates more manpower to production and less to innovation, leading to lower growth rates in the long run. This is consistent with our intuition: Inventors only internalise that innovating is profitable for the sake of reselling the patent. They ignore the externality conferred on future inventors, who get to stand on the shoulder of giants: $A' = A + \eta A(1-\phi)H$, where the second A on the right-hand side is the innovation externality.

(c) **Solve for the growth rate of A, $g_A = \frac{A_{t+1}}{A_t}$, along the balanced growth path.**

From the previous step, we obtain

$$\begin{aligned}\gamma_A^{SP} &= \frac{A'}{A} = 1 + \eta(1-\phi^{SP}) \\ &= 1 + \frac{\eta H\beta - 1}{H(1+\beta)}\end{aligned}$$

Comparing that with $\gamma^{CE} = \frac{\beta(\alpha+\eta H\alpha+1)}{1+\beta\alpha}$, we see that even with no externality ($\eta = 0$) there is a discrepancy in growth rates:

$$\begin{aligned}\gamma_A^{SP} &= \beta \\ \gamma_A^{CE} &= \frac{\beta(\alpha+\eta H\alpha+1)}{1+\beta\alpha} \\ \implies \gamma^{CE} &= \gamma^{SP}\frac{\alpha+\eta H\alpha+1}{1+\beta\alpha}\end{aligned}$$

I suspect that this difference comes from the fact that intermediate goods producer in CE exert market power, distorting output, and hence also intertemporal considerations. This also brings us away from first-best.

(d) **Assume that $\gamma = 0$. Along the balanced growth path, what is the equilibrium increase in varieties each period, i.e. $A_{t+1} - A_t$? What is the equilibrium growth rate of A in this case? Briefly explain the intuition for your answer.**

With $\gamma = 0$, our previous calculations are changed in two spots: The Euler Equation w.r.t. $[\phi]$ and the law of motion for A. Re-doing that part, and imposing a BGP where we again conjecture that $\phi' = \phi$:

$$\begin{aligned}[\phi] : \beta\frac{\partial}{\partial A'}V(A',k')\cdot\eta H &= \lambda(1-\alpha)\phi^{-\alpha}(AH)^{1-\alpha}k^{\alpha} \\ \implies \beta\big[(\phi' H)\Big(\frac{k'}{A'}\Big)^{\alpha}\big]\eta &= \frac{c'}{c}A^{1-\alpha}k^{\alpha} \\ \overset{BGP}{\implies} \beta\phi H\eta &= \gamma_c\Big(\frac{\gamma_A}{\gamma_k}\Big)^{\alpha}A \\ [EE - k'] \longrightarrow \gamma_k = \gamma_A \implies \beta\phi H\eta &= \gamma_c A \\ &\implies A \text{ const.}\end{aligned}$$

which implies long-run technological stagnation. And confirm the initial conjecture using $[LoM]$:

$$A' = A + \eta(1-\phi)H \implies \frac{A'}{A} = \gamma_A = 1 + \frac{\eta(1-\phi)H}{A}$$
$$\implies \phi \text{ const. and } \phi^{SP} = 1$$

i.e. zero research effort, consistent with technological stagnation.

Note that this is the planner's solution. We arrive at this dire conclusion not because inventors ignore the externality that future inventors are standing on their shoulders. This is socially efficient. The intuition is that research becomes exceedingly costly compared to consumption, so $1-\phi_t \to 0$ along the transition path.

One may see that this result also holds for any $\gamma < 1$, such as $\gamma = 0.9999$. Why do we have such a knife-edge property? The intuition is as follows: For positive balanced growth, we need that A grows at some positive rate. But $A' - A$ becomes linear for $\gamma = 0$: $A' - A = \eta(1-\phi)H$. That means, as we require $A' - A$ to become ever larger to keep up constant growth, we would need to dedicate more and more hours to research. As PhD students you may have experienced that at some point there is a hard limit of zero sleep, or here $\phi = 0$. We simply physically won't be able to keep up. In the $\gamma = 1$ case, $A' - A = \eta(1-\phi)H \cdot A$, so we could scale with the level and put in constant hours.

Question 4: Overlapping Generations Model with Social Security

Consider the following overlapping generations model. The economy is populated by measure one of households that live for two periods. Households are endowed with one unit of labor when young, which they supply inelastically, and no time endowment when old. The preferences of an agent born at time t are:

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t),$$

where c_t^t and c_{t+1}^t are consumption when young and when old, and $0 < \beta < 1$. Households save capital k_{t+1}^t when young and earn rental income from that capital when old. The rental rate at t is denoted r_t, and the wage rate is w_t. The production technology is $Y_t = AK_t^\theta N_t^{1-\theta}$, where K_t and N_t are aggregate capital and labor inputs, $0 < \theta < 1$, and A is a positive constant. Markets are perfectly competitive, and the production technology is operated by competitive firms. Capital depreciates fully each period.

The government runs a "pay as you go" social security system that takes d_t units of the final good from the young in period t and redistributes it to the old in period t in the form of a benefit, b_t, where $d_t = b_t$. Assume that the entire sequence of transfers, $\{d_t\}_{t=0}^{\infty}$, is known to all agents and feasible (meaning the transfer is less than wage income for the young in each period). There is an initial old generation that is endowed with capital k_0^{-1}.

(a) **Write the household's problem and firm's problem and characterize the optimality conditions for the household and firm. Define a sequence-of-markets equilibrium.**

A household born in period t takes the sequence of prices and net transfers as given: $\{w_t, r_t, d_t\}_{t=0}^{\infty}$ and solves:

$$V_t = \max_{c_t^t, c_{t+1}^t, k_{t+1}} \log(c_t^t) + \beta \log(c_{t+1}^t) \quad s.t.$$

$$\begin{cases} c_t^t + k_{t+1} = w_t - d_t \\ c_{t+1}^t = r_{t+1} k_{t+1} + d_{t+1} \end{cases}$$

Optimality conditions are:

$$\begin{aligned}
\mathcal{L} = {} & \log(w_t - d_t - k_{t+1}) + \beta \log(r_{t+1} k_{t+1} + d_{t+1}) \\
[k_{t+1}] : & \frac{1}{w_t - d_t - k_{t+1}} = \beta \frac{r_{t+1}}{w_t - d_t - k_{t+1}} \\
\Longrightarrow & \, r_{t+1} k_{t+1} + d_{t+1} = \beta r_{t+1} (w_t - d_t - k_{t+1}) \\
[BC1] : & \, c_t^t + k_{t+1} = w_t - d_t \\
[BC2] : & \, c_{t+1}^t = r_{t+1} k_{t+1} + d_{t+1}
\end{aligned}$$

Firms solve their usual profit objective, taking prices as given:

$$\max_{K_t, N_t} A_t K_t^{\theta} N_t^{1-\theta} - r_t K_t - w_t N_t$$

$$[K_t] : A_t \theta \Big(\frac{K_t}{N_t}\Big)^{\theta - 1} = r_t$$

$$[N_t] : A_t (1-\theta) \Big(\frac{K_t}{N_t}\Big)^{\theta} = w_t$$

The competitive equilibrium is defined as:

(a) Household choices $\{c_t^t, c_{t+1}^t, k_{t+1}\}_{t=0}^{\infty}$

(b) Firm factor demand $\{k_t, N_t\}_{t=0}^{\infty}$

(c) Sequences of prices $\{w_t, r_t\}_{t=0}^{\infty}$

such that

(i) Given (3), households solve their household programme, yielding (1)

(ii) Given (3), firms solve their profit maximisation programme, yielding (2)

(iii) Markets clear $\forall t$:

- Goods: $c_t^t + c_t^{t-1} + k_{t+1} = A_t k_t^\theta N_t^{1-\theta}$
- Labour market: $N_t = 1$
- Capital market: $K_t = k_t$

(I ignored the fact that c_0^{-1} is not well-defined, but let's be real nobody wants to change notation just for the initial period...)

(b) **Suppose the social-security benefit is a constant value, $\bar{b}$, each period, equal to a fraction γ times the steady-state capital stock, where $0 < \gamma < 1$. Solve for the steady-state capital stock, consumption of the young, and consumption of the old.**

Gathering all the optimality conditions:

$$\begin{aligned}
[EE] &: \frac{1}{c_t} = \beta \frac{r_{t+1}}{c_{t+1}} \\
[BC1] &: c_t^t = w_t - d_t - k_{t+1} \\
[BC2] &: c_{t+1}^t = r_{t+1} k_{t+1} + d_{t+1} \\
[K_t] &: r_t = A_t \theta \Big(\frac{K_t}{N_t} \Big)^{\theta - 1} \\
[N_t] &: w_t = A_t (1-\theta) \Big(\frac{K_t}{N_t} \Big)^{\theta} \\
[MCc] &: c_t^t + c_t^{t-1} + k_{t+1} = A_t K_t^\theta N_t^{1-\theta} \\
[MCN] &: N_t = 1 \\
[MCK] &: K_t = k_t
\end{aligned}$$

Imposing steady state and our specification of $d_t = \gamma k$:

$$\begin{aligned}
[EE] &: \frac{1}{c^1} = \beta \frac{r}{c^2} \\
[BC1] &: c^1 = w - \gamma K - K \\
[BC2] &: c^2 = rK + \gamma K \\
[k] &: r = A\theta k^{\theta - 1} \\
[N] &: w = A(1-\theta) K^\theta \\
[MCc] &: c^1 + c^2 + k = Ak^\theta
\end{aligned}$$

Plugging everything into the Euler equation:

$$\frac{1}{w-(1+\gamma)K} = \beta\frac{r}{(r+\gamma)k} \implies (r+\gamma)K = \beta r(w-(1+\gamma)K)$$
$$\implies (A\theta K^{\theta-1}+\gamma)K = \beta A\theta K^{\theta-1}\Big(A(1-\theta)K^{\theta}-(1+\gamma)K\Big)$$

This can be simplified a bit to yield a quadratic equation in $K^{\theta-1}$:

$$\beta A^2\theta(1-\theta)K^{2(\theta-1)} - (1-\beta(1+\gamma))A\theta K^{\theta-1} - \gamma = 0$$

Which can be solved numerically given parameters.

(c) **Suppose** $\beta = 0.9$**,** $\gamma = 0.1$**,** $A = 1$**,** $\theta = 0.3$. **Does the presence of the "pay as you go" social security system increase consumption relative to the steady state without social security? Provide a brief intuition for your results. You may want to draw on the solution to a planner's problem.**

Before plugging specific values into the computer, we can already see that γ shows up in two coefficients of the quadratic equation and hence affects the value of steady-state capital.

In the following plot I show the consumption levels for different rates of γ. We can see that the young consume **more** when social security contributions increase up to roughly $\gamma = 0.1$, even though they are thxe ones being taxed!

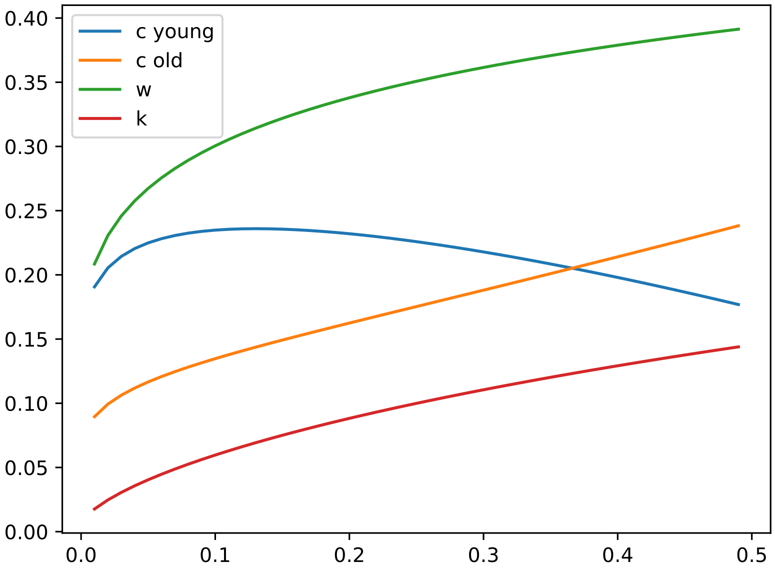

Figure 1: consumption of the generations, for different social scurity contribution rates