Problem Set #2 Instructor: David Laibson
Due Sunday, September 15, 10pm Economics 2010c

Problem 1 (Eat-the-Pie Problem): Consider the following sequence problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(c_t)$$

subject to the constraints:

$$W_{t+1} = R(W_t - c_t)$$

$$0 \leq c_t \leq W_t$$

$$W_0 > 0 \quad \text{given.}$$

I'm now going to ask you to analyze this problem. This analysis provides a quick review of concepts that should by now be familiar. I will put problems like this on the final exam.

a. Motivate the economic problem above. Evaluate the implicit assumptions. What is economically sensible and what is not sensible about this modeling set-up?

b. Explain why the Bellman equation for this problem is given by:

$$v(W) = \sup_{c\in[0,W]} \{u(c) + \delta v(R(W-c))\} \quad \forall W$$

Why doesn't an expectation operator need to appear in this Bellman equation?

c. Using Blackwell's sufficiency conditions, prove that the Bellman operator, B,

$$(Bf)(W) = \sup_{c\in[0,W]} \{u(c) + \delta f(R(W-c))\} \quad \forall W$$

is a contraction mapping. You should assume that u is a bounded function. (Why is this boundedness assumption necessary for the application of Blackwell's Theorem?) Explain what the contraction mapping property implies about iterative solution methods.

d. Now assume that,

$$u(c) = \left\{ \begin{array}{ll} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \ln c & \text{if } \gamma = 1 \end{array} \right\}.$$

(So u is no longer bounded.) Use the guess method to solve the Bellman equation. Specifically, guess the form of the solution:

$$v(W) = \left\{ \begin{array}{ll} \psi \frac{W^{1-\gamma}}{1-\gamma} & \text{if } \ \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \phi + \psi \ln W & \text{if } \ \gamma = 1 \end{array} \right\}.$$

Derive the optimal policy rule:

$$c = \psi^{-\frac{1}{\gamma}} W$$

$$\psi^{-\frac{1}{\gamma}} = 1 - (\delta R^{1-\gamma})^{\frac{1}{\gamma}}$$

Note that this rule applies for all values of γ. Confirm that this solution to the Bellman Equation works.

e. When $\gamma = 1$ the consumption rule collapses to $c_t = (1-\delta)W_t$. Why does consumption no longer depend on the value of the interest rate (for a given W_t)? Hint: think about income effects and substitution effects.

Problem 2 (Search and Optimal Stopping): Reconsider the optimal stopping problem:

Each period (over an infinite horizon) the consumer draws a job offer from a uniform distribution with support in the unit interval: $x \sim u[0,1]$. *The consumer can either accept the offer and realize NPV* x, *or the consumer can wait another period and draw again. Once you accept an offer the game ends. Waiting to accept an offer is costly because the value of the remaining offers declines at rate* $\rho = -\ln \delta$ *between periods. The Bellman equation for this problem is:*

$$v(x) = \max\{x, \delta E v(x_{+1})\}$$

a. Explain the intuition behind the Bellman Equation.

b. Consider the associated functional operator:

$$(Bw)(x) \;\; = \;\; \max\{x, \delta E w(x_{+1})\} \;\; \forall x.$$

Using Blackwell's conditions, show that this Bellman operator is a contraction mapping.

c. What does the contraction property imply about $\lim_{n\to\infty} B^n w$, where w is an arbitrary function?

d. Let $w(x) = 1\ \forall x$. *Analytically*, iterate $B^n w$, and show that

$$\lim_{n\to\infty} (B^n w)(x) = v(x) = \left\{ \begin{array}{lll} x^* & \text{if} & x \leq x^* \\ x & \text{if} & x > x^* \end{array} \right\}$$

where,

$$x^* = (\exp \rho)\left(1 - [1 - \exp(-2\rho)]^{1/2}\right).$$

Hint: use a similar conceptual approach to the one that we used in section when $w = 0$.

Problem 3: Consider an **optimal investment** problem.

- Every period you draw a cost c (distributed uniformly between 0 and 1) for completing a project.
- If you undertake the project, you pay c, and complete the project with probability $1 - p$.
- At the beginning of each period in which the project remains uncompleted, you pay a late fee of l.
- The game continues until you complete the project.

a. Write down the Bellman Equation assuming no discounting. Why is it OK to assume no discounting in this problem?

b. Derive the optimal threshold: $c^* = \sqrt{2l}$. Explain intuitively, why this threshold does not depend on the probability of failing to complete the project, p.

c. How would these results change if we added discounting to the framework? Redo steps a and b, assuming that the agent discounts the future with discount factor $0 < \delta < 1$ and assuming that $p = 0$. Show that the optimal threshold is given by.

$$c^* = \frac{\delta - 1 + \sqrt{(1-\delta)^2 + 2\delta^2 l}}{\delta}.$$

d. When $0 < \delta < 1$, is the optimal value of c^* still independent of the value of p? If not, how does c^* qualitatively vary with p? Provide an intuitive argument.

e. Prove that the expected delay until completion is given by:

$$[c^*(1-p)]^{-1} - 1.$$

Note that this calculation takes the perspective on an agent who hasn't yet observed the current period's draw of c. So there is a $c^*(1-p)$ probability that the delay is 0.

Problem 4 (Coding) In this problem, we ask you to solve a Bellman equation numerically using Value Function Iteration. You can use any programming language you prefer, but solutions will be provided in Julia. Please submit a copy of your code with the assignment.

A consumer has log utility over consumption (c) and a discount rate δ. Wealth x evolves stochastically according to:

$$x_{+1} = x - c + \tilde{y}_{+1}$$

(So the gross rate of return, R, is equal to 1.) Here $\tilde{y}_{+1}$ is i.i.d. stochastic labor income. Note, therefore, that x_{+1} is defined as total wealth, including labor income. As a result, we do not have to keep track of $\tilde{y}$ as a state variable. (Optional: what would happen if $\tilde{y}$ weren't i.i.d.?)

Define the *savings* policy of the investor as $s = x - c$. We can write the Bellman equation as:

$$V(x) = \max_{s \in [0,x]} \left\{ \log(x - s) + \delta E V(s + \tilde{y}_{+1}) \right\}$$

We will now walk you through coding up value function iteration. Problem 5 of the previous problem set has additional tips and hints that will be very useful for this exercise.

1. Specify the parameter values
   (a) $\delta = 0.9$
   (b) $\tilde{y}_{+1}$ is distributed uniformly on $Y = \left[\frac{1}{10}, \frac{2}{10}, ..., \frac{9}{10}\right]$

(c) The wealth level is on a grid $X = [0.01, 0.02, ..., 10]$

2. To start, let us take one iteration of the Bellman equation. Start from an initial guess $V(x_{+1}) = x_{+1}$ at every point on your grid X.

   (a) Solve for the continuation value function (i.e. $EV(s+\tilde{y}_{+1})$), given inherited wealth s. The following subparts walk you through this process:

      i. Create two matrices which encode the set of pairs $(s, y) \in X \times Y$.[1] One matrix should have repeated values of X, and the other should have repeated values of Y, so that the dimensions of each matrix is $|X| \times |Y|$ (or $|Y| \times |X|$, if you prefer). (Note: *repeat* in Julia will be a useful function.)

      ii. For each pair (s, y), add the two to get a a value x_{+1}.

      iii. Find the associated value $V(x_{+1})$ from your initial grid of guesses $V(X)$. You can either find points in the grid or use interpolation. Don't use the analytic form, since when you iterate you will no longer have an analytic representation. You can deal with values of x_{+1} higher than 10 by either truncating them (i.e. assuming that $V(x_{+1}) = V(10)$) or using extrapolation (see Problem 6 of the last problem set if you're interested).

      iv. For each s, take the expectation over y to get $V^C(s) \equiv E\left[V(x_{+1})|s\right]$. Note that the expectation operator is very simple, since we have discretized the income shock.

   (b) Now, you have each savings value s associated with a continuation value $V^C(s)$. We want to solve now for the value function $V(x)$.

      i. Create two matrixes for every pair $(x, s) \in X \times X$ of initial wealth and savings level (i.e. one matrix with x in the rows, another one with s in the columns). Do not worry about the fact that you have points $s > x$ (we will handle that later).

      ii. Using the continuation value vector you found above, create a matrix V^C that corresponds to the continuation values $V^C(s)$ for each cell in your matrix.

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[1] If you need more details, check how we defined k_vals and y_vals last Problem Set.

iii. Create a flow utility matrix U for each (x, s). You can construct this by:

$$U = \log\left(\max\{x - s, \underline{c}\}\right)$$

Where $\underline{c}$ is a parameter we set to be very small to ensure the log is well defined. For example, set $\underline{c} = 10^{-10}$. Because this will generate such high flow disutility, the optimum will never feature $s \geq x$. In other words, $\underline{c}$ enforces our constraint.

iv. Add together the flow and continuation utilities to get a matrix $V = U + \delta V^C$ of values, for each pair (x, s).

v. Now, for each x, take the maximum value of V over s. This gives us our new value function, defined over each point X of the grid.

vi. Optional, for people that like Macro: You can also solve this by using interpolation. For each value of x, define a function $f(s) = \log(x - s) + \delta V^C(s)$, and find the maximum using either bisection or a built-in optimizer.

3. Part 2 walked you through one iteration of value function iteration. Now, all that there remains to do is to iterate. Create a loop that takes a guess V_1 for the value function, applies the steps of part 2 to derive a new value function V_0 from the Bellman equation, and then replaces $V_1 = V_0$ and starts the loop over again. Note that you will need to define a tolerance level (try 1e-8), and check the distance between V_1 and V_0 in each step (*norm* can do this for you).

   (a) Iterate from the starting guess $V_1(x) = x$ until the your estimated V changes less than the tolerance level between iterations.

   (b) Plot the value function against x.

   (c) Plot the optimal policy function s against x.

   (d) Start from a different guess, $V_1(x) = 0$, and confirm that you converge to the same value function.

With this, you have solved value function iteration numerically. For this part of the problem set, submit the plots from 3b, 3c, and 3d.

Problem 5 (optional): Problem based on the Diamond-Mortensen-Pissarides model (thanks to Ben Hebert and Argyris Tsiaras)

This problem will take you through the math behind the Diamond-Mortensen-Pissarides model, a search and matching model that explains some aspects of unemployment and labor dynamics. Those three won the Nobel prize in 2010 for the development of this model, among other contributions. The specific model discussed below is an adaption of a 2005 AER paper by Robert Shimer.

Assume that every actor in the problem below uses a discount factor δ and is risk-neutral (which means that their flow utility is linear in money).

0.1 The Filled Job Problem

For an employer, when they have a job that is filled, they earn p_t each period, but must pay the wage w. The notation p_t implies that the earnings change over time, but the wages paid, w, are constant. Assume that p_t is stochastic, but becomes known at the beginning of each period.

With probability s, the job ends ("separates"), and this employer gets zero forever after. Separations are exogenous, meaning that they do not depend on p_t, w, or anything else. For each worker-employer pair, they just separate or don't separate, each period, with the same constant probability. This is not a choice that either the worker or the employer makes; it just happens or doesn't. Separations happen after the employer earns money and the worker is paid.

Denote the employer's filled job value function as $J(p_t, w)$.

Problem: Write down the Bellman equation for $J(p_t, w)$. What is $\frac{\partial J(p_t,w)}{\partial w}$?

Hint: Are there any choices in this problem? To derive the comparative static, think about the corresponding sequence problem.

0.2 The Employed Worker's Problem

Assume that every period, employed workers earn a wage w. With probability s, after earning their wage, they end employment and become unemployed. With probability $1-s$, they remain employed. Again, separation is exogenous and not a choice. Denote the employed worker's value function as $W(p_t, w)$, and the unemployed worker's value function as $U(p_t)$.

Problem: Write down the Bellman equation for employed workers. What is $\frac{\partial W(p_t,w)}{\partial w}$?

Hint: Does the wage w affect the worker's unemployment utility? Think about the sequence problem again.

0.3 The Matching Function

Define the number of unemployed workers as u_t, and the number of job openings (vacancies) as v_t. The ratio of the two, $\theta_t = \frac{v_t}{u_t}$, is going to be an important state variable. Define the number of "matches" (worker-vacancy pairs) that occur each period as

$$m(u_t, v_t) = \kappa u_t^{\alpha} v_t^{1-\alpha}$$

Problem: Assume that all unemployed workers have the same probability of filling a given vacancy. For an individual unemployed worker, what is the probability of being matched, given θ_t? For an employer with one vacancy, what is the probability of finding an employee, given θ_t?

Hint: the two probabilities depend only on θ_t and constants.

0.4 The Unemployed Worker's Problem

Assume the unemployed workers can earn z while unemployed, representing the value of leisure, non-market production, unemployment benefits, and the like. They get z at the start of the period, and then either find a job or don't. If they find a job, it starts in period $t+1$, with a wage $w(p_t)$.

Denote the unemployed worker's value function as $U(p_t)$. Assume that θ_t is a function of p_t, and write $\theta(p_t)$.

Problem: Write down the Bellman Equation for the value function for an unemployed worker.

Hint: Use the probability of finding a job you solved for earlier.

0.5 The Surplus Bellman Equation

Define the surplus of an existing match, given a wage $\hat{w}$, as

$$S(p_t, \hat{w}) = W(p_t, \hat{w}) - U(p_t) + J(p_t, \hat{w}).$$

Assume that w is chosen by Nash bargaining when the match is made, so that

$$w(p_t) = argmax_{\hat{w}} (W(p_t, \hat{w}) - U(p_t))^{\beta} J(p_t, \hat{w})^{1-\beta}.$$

Define the surplus of a new match

$$V(p_t) = S(p_t, w(p_t))$$

with w as the Nash bargaining solution.

Problem: Derive the equation below. You can assume that the surplus is always positive.

$$V(p_t) = p_t - z - \delta(\beta\kappa\theta(p_t)^{1-\alpha} - 1 + s)E_t[V(p_{t+1})].$$

Hint: Look at the FOC for the Nash bargaining problem. Then write out $V(p_t)$ using W, U, and J. Why does the wage not appear in the Bellman Equation for V?

0.6 The Free Entry Condition

Assume that the cost of posting a vacancy is c per period, and that there is free entry of employers, so that the cost per period is always equal to the expected benefit.

Problem: Using the free entry assumption, derive the following equation:

$$\frac{c\theta(p_t)^\alpha}{\kappa(1-\beta)} = p_t - z - \frac{c\delta(\beta\kappa\theta(p_t)^{1-\alpha} - 1 + s)}{\kappa(1-\beta)}E_t[\theta(p_{t+1})^\alpha].$$

Hint: The expected benefit to employers of posting a vacancy is the probability of filling it, multiplied by the lifetime benefit of doing so. Use the FOC of the Nash bargaining equation to relate J and V.

0.7 Comparative Statics

Problem: Assume that $E[\theta(p_{t+1})^\alpha] = \theta(p_t)^\alpha$ (there is no uncertainty or change between t and $t+1$). Compute the elasticity of the vacancy-unemployment ratio $\theta(p_t)$ with respect to net labor productivity $p_t - z$, around the value of $\theta(p_t) = 1$. That is,

$$(p_t - z)\frac{d\theta(p_t)}{d(p_t - z)}|_{\theta(p_t)=1} =?$$

Hint: your expression should contain only the constants listed below. Don't forget that $\theta(p_t) = 1$ implies something about $p_t - z$, and you should use this to eliminate $p_t - z$ terms from your expression.

0.8 Calibration

Use the following values, which are calibrated to quarterly data:

$$s = 0.034$$

$$\delta = 0.996$$

$$\alpha = 0.72$$

$$\kappa = 0.45$$

$$\beta = 0.72$$

Problem: Evaluate the elasticity in the previous part under this calibration. Use a loglinear approximation of θ_t (as a function of net labor productivity, $p_t - z$) around its deterministic steady state characterized by $\theta_t = 1$ for all t to arrive at the model's prediction for the ratio of the standard deviation of the log vacancy-unemployment ratio, $\sigma(\log \theta_t)$, to the standard deviation of log net labor productivity, $\sigma\left(\log\left(p_t - z\right)\right)$. In the actual data, this ratio is estimated to be above 10 (much higher than the ratio that you will find). This is the point that Shimer made: fluctuations in worker productivity, in this version of the DMP model, cannot explain the observed fluctuations in vacancies and unemployment.