

ECON 2010c

Solution to Problem Set 2*

Fall 2024

1 Eat-the-Pie Problem

a. The consumer selects consumption in each period, given her current wealth, which equals savings in the previous period times the gross rate of return. The consumer has already sold any claims to future labor income and cannot consume an amount greater than her current wealth. The following are some implicit assumptions of the model:

- The consumer knows the rate of return on savings with certainty, eliminating a possible motive for precautionary savings.
- The consumer is assumed to be able to sell claims to all their future labor income for their expected discounted value. Under this interpretation the assumption that no borrowing is allowed is natural (it follows from a no-Ponzi condition).
- The consumer is assumed to discount the future exponentially and so have time-consistent preferences, even though there is some evidence indicating that consumers find it difficult to adhere to their previous consumption plans.
- The utility function is additively separable across time periods; for example, the model does not consider the possibility of habit formation.
- The flow payoff function does not change over time or across states. That is, the consumer does not have time-varying or state-dependent preferences.
- The consumer is assumed to face an infinite-horizon decision problem. We might think of three ways of justifying this assumption:
 - Consumers are altruistic towards future generations of their families
 - Consumers are far away from death, and they can approximate their current optimal consumption/savings well by using the infinite horizon problem (more to come in Lecture 5)
 - Consumers do not know when they will die, and the discounting δ^t reflects partly the probability that a consumer will have died by period t

*. These answers are based on the previous work of Argyris Tsiaras, Filipe Campante, Fabià Gumbau, Jason Hwang, Yves Nosbusch, Neel Rao, Jenny Tang, Ben Hebert, and particularly Davide Cantoni and Giacomo Ponzetto.

- b. The Bellman equation is as claimed in the problem set, because the value of current wealth $v(W)$ is equal to the flow payoff from current consumption $u(c)$ plus the discounted value of future wealth $\delta v(R(W - c))$, where current consumption c is chosen to maximize the sum of the latter two terms. The expectation operator is not needed because the consumer is not exposed to any uncertainty (the stochastic income stream was already sold for a constant, known amount of assets and the return rate on savings is known with certainty).
- c. First, note that we assume that the domain of B is the space of bounded functions. We can restrict our attention to this space because the fact that u is bounded implies that the value function for our sequence problem

$$v^{SP}(W_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(c_t)$$

is also bounded. To see this, note that if $K > 0$ is an upper bound for $u(x)$ then $\frac{K}{1-\delta}$ is an upper bound for $v^{SP}(W_0)$, so that the latter is also bounded. Since our ultimate goal is to compute the solution to the sequence problem, we can restrict our attention to solutions $v(x)$ of the Bellman equation that are bounded. Also note that, if u is bounded, the Bellman operator B will always map a bounded function f to another bounded function Bf , since the latter is the sum of two bounded functions, $u(\cdot)$ and $\delta f(\cdot)$. Therefore, B maps the space of bounded functions into itself, $B : C(X) \rightarrow C(X)$, as Blackwell's Theorem requires.

Then, we can apply Blackwell's theorem, which provides sufficient conditions for a mapping from the space of bounded functions into itself to be a contraction. We now show that our Bellman operator B satisfies Blackwell's sufficient conditions. The monotonicity condition holds, because the following is true for functions f and g such that $f \leq g$:

$$Bf(W) = \sup_{c \in [0, W]} \{u(c) + \delta f(R(W - c))\} \leq \sup_{c \in [0, W]} \{u(c) + \delta g(R(W - c))\} = Bg(W),$$

where the inequality is a straightforward consequence of the fact that $f(W) \leq g(W)$ for each W . Provided that $\delta \in (0, 1)$, the discounting condition holds, because the following is true for a function f and a constant $\alpha \geq 0$:

$$\begin{aligned} B(f + \alpha)(W) &= \sup_{c \in [0, W]} \{u(c) + \delta [f(R(W - c)) + \alpha]\} \\ &= \sup_{c \in [0, W]} \{u(c) + \delta f(R(W - c))\} + \delta \alpha = Bf(W) + \delta \alpha. \end{aligned}$$

Because the Bellman operator B is a contraction mapping, the contraction mapping theorem implies that the repeated iteration of B on a starting function w generates a sequence of functions that converges to the unique fixed point v of the Bellman operator, which is a bounded function, thus providing the solution to the Bellman equation.

- d. The first-order condition and the envelope theorem yield the following:

$$u'(c) = \delta R v'(R(W - c)) = v'(W).$$

Given the form of the utility function and the value function being assumed, we have $u'(c) = c^{-\gamma}$ and $v'(W) = \psi W^{-\gamma}$ which is valid for any $\gamma \in (0, \infty)$. Substituting for $u'(c)$ and $v'(W)$ in the previous expression yields:

$$c^{-\gamma} = \delta R \psi [R(W - c)]^{-\gamma} = \psi W^{-\gamma} \Rightarrow c = \psi^{-\frac{1}{\gamma}} \delta^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} (W - c) = \psi^{-\frac{1}{\gamma}} W,$$

confirming that the optimal policy rule is $c = \psi^{-\frac{1}{\gamma}}W$. The constant ψ can be obtained from the following:

$$\begin{aligned}\psi^{-\frac{1}{\gamma}}\delta^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}\left(W - \psi^{-\frac{1}{\gamma}}W\right) &= \psi^{-\frac{1}{\gamma}}W \Rightarrow \left(1 - \psi^{-\frac{1}{\gamma}}\right)\delta^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}W = W \\ \Rightarrow 1 - \psi^{-\frac{1}{\gamma}} &= \delta^{\frac{1}{\gamma}}R^{\frac{1}{\gamma}-1} \Rightarrow \psi^{-\frac{1}{\gamma}} = 1 - \left(\delta R^{1-\gamma}\right)^{\frac{1}{\gamma}}.\end{aligned}$$

In the case where $\gamma = 1$, we have $\psi = (1 - \delta)^{-1}$ and $c = (1 - \delta)W$; so that, the constant ϕ can be recovered as follows:

$$\begin{aligned}\phi + \psi \ln W &= \sup_{c \in [0, W]} \{ \ln c + \delta \{ \phi + \psi \ln [R(W - c)] \} \} \\ \Rightarrow \phi + \frac{1}{1 - \delta} \ln W &= \ln(1 - \delta) + \ln W + \delta \left\{ \phi + \frac{1}{1 - \delta} [\ln(\delta R) + \ln W] \right\} \\ \Rightarrow \phi &= \frac{1}{1 - \delta} \left[\ln(1 - \delta) + \frac{\delta}{1 - \delta} \ln(\delta R) \right],\end{aligned}$$

which confirms that the solution indeed satisfies the Bellman equation for $\gamma = 1$.

In the case where $\gamma \in (0, \infty)$ and $\gamma \neq 1$, the Bellman equation is satisfied because:

$$\begin{aligned}\psi \frac{W^{1-\gamma}}{1 - \gamma} &= \sup_{c \in [0, W]} \left\{ \frac{c^{1-\gamma}}{1 - \gamma} + \delta \psi \frac{[R(W - c)]^{1-\gamma}}{1 - \gamma} \right\} \\ \Rightarrow \psi \frac{W^{1-\gamma}}{1 - \gamma} &= \frac{\left(\psi^{-\frac{1}{\gamma}}W\right)^{1-\gamma}}{1 - \gamma} + \delta \psi \frac{[R(W - \psi^{-\frac{1}{\gamma}}W)]^{1-\gamma}}{1 - \gamma} \\ \Rightarrow 1 &= \psi^{-\frac{1}{\gamma}} + \delta \left[R \left(1 - \psi^{-\frac{1}{\gamma}}\right) \right]^{1-\gamma} \Rightarrow \psi = \left(1 - \delta^{\frac{1}{\gamma}}R^{\frac{1}{\gamma}-1}\right)^{-\gamma}.\end{aligned}$$

- e. First, it is important to understand that the consumption path is *always* responsive to interest rate changes. This follows immediately from the loglinearized Euler Equation

$$\Delta \ln c_{t+1} = \frac{1}{\gamma}(r - \rho)$$

which states that a one-percent increase in the gross simple interest R (that is, an one-unit level increase in $r \equiv \ln R$) will increase consumption growth by $\frac{1}{\gamma}$ percentage points. For example, for $\gamma = 1$, the growth rate of the consumption path increases by one percentage point in response to a one-percent increase in R . Another way to see this point is that W_t depends on R , for all $t > 0$, so that $c_t = (1 - \delta)W_t$, for $t > 0$, also changes with R .

When $\gamma = 1$, consumption does not depend *directly* on the gross rate of return R (as explained in the previous paragraph, it depends indirectly on R through W_t), because the income and substitution effects of an interest rate change on consumption cancel each other out.

In general, the substitution effect of a price change on the demand for a good i is the impact of the (marginal) price change on good i 's demand, *holding purchasing power fixed*. This is formally defined to be the demand change of a consumer who has received the *Slutsky wealth compensation*

associated with the price increase (see MWG, ch. 2.F). The income effect of the price change is the effect of the price change on the consumer's purchasing power. It is formally defined to be the demand impact of an increase in the agent's wealth by an amount equal to the negative of the Slutsky wealth compensation, holding the price fixed at its initial level.

I will present a formal treatment of substitution and income effects in Section 4. Here, I give an informal discussion (which is all that is required for this question).

We can intuitively understand the substitution effect of an interest rate increase by looking at the Euler equation, which gives us the impact of an interest rate increase on the consumption growth of a CRRA agent. A marginal proportional increase in R , dr , leads to a proportional increase in consumption growth equal to $\frac{1}{\gamma}dr$. To informally capture the effect of the interest rate increase holding purchasing power constant, let's assume that c_{t+1} is fixed at its initial value. Then, the Euler Equation tells us that consumption today should fall by $\frac{1}{\gamma}dr$ percentage points. Intuitively, the interest rate is the price (opportunity cost) of current consumption relative to future consumption. Therefore, current consumption, c_t , has become more expensive, which gives the agent an incentive to reduce his demand for c_t .

The income effect on c_t of the interest rate increase is positive, corresponding to the higher purchasing power of the agent's wealth, since the amount of consumption that the agent can attain for any given level of savings increases. Since consumption at any time period is a normal good, its demand tends to increase as a result of the increase in purchasing power. That is, the income effect corresponds to an incentive to increase consumption at all points in time. To informally capture its relative magnitude, we hold constant the level of time t savings, \bar{y}_t , and note that, for the CRRA utility, consumption c_{t+1} is homogeneous in (proportional to) that period's wealth $W_{t+1} = R\bar{y}_t$, so that the income effect of a proportional interest rate increase dr is associated with an increase in c_{t+1} by dr percentage points.

We see that, for a CRRA agent, the substitution and income effects lead to a change in c_t by $-\frac{1}{\gamma}dr$ and dr percentage points, respectively. In the $\gamma = 1$ case, the two effects cancel out.

2 Search and Optimal Stopping

- a. In each period, the consumer decides whether to accept the current offer of x or to continue searching for another offer. If the consumer accepts an offer of x , then she receives a payoff of x . If the consumer rejects the offer, then she faces an analogous problem next period, drawing an offer x_{+1} from the same distribution. Noting that x_{+1} is unknown in the current period and $\rho = -\ln \delta$ is the rate at which the future is discounted, $\delta Ev(x_{+1})$ is the continuation value from rejecting an offer. Thus, the value of drawing an offer of x is the greater of the payoff x from accepting the offer and the value $\delta Ev(x_{+1})$ from continuing to search.
- b. Note that B is a mapping from the space of bounded functions into itself, since the sup norm of $Bw(x)$ is equal to the maximum of 1 and δ times the sup norm of $w(x)$, which is finite if $w(x)$ is bounded. Thus, we can apply Blackwell's Theorem and show that the Bellman operator B is a contraction mapping, because it satisfies Blackwell's sufficiency conditions. The monotonicity condition holds, because the following is true for functions f and g with $f \leq g$:

$$Bf(x) = \max \{x, \delta Ef(x_{+1})\} \leq \max \{x, \delta Eg(x_{+1})\} = Bg(x),$$

where the inequality results from $Ef(x_{+1}) \leq Eg(x_{+1})$. Provided that $\rho > 0$, the discounting condition holds, because the following is true for a function f and a constant $\alpha \geq 0$:

$$\begin{aligned} B(f + \alpha)(x) &= \max\{x, \delta E[f(x_{+1}) + \alpha]\} \\ &\leq \max\{x, \delta E[f(x_{+1})]\} + \delta\alpha = Bf(x) + \delta\alpha \end{aligned}$$

- c. Because the Bellman operator B is a contraction mapping from a complete metric space into itself, the contraction mapping theorem implies that $\lim_{n \rightarrow \infty} B^n w = v$ for any arbitrary function w , where v is the unique fixed point of the Bellman operator and thus the solution to the Bellman equation.
- d. Let $w(x) = 1$ for all $x \in [0, 1]$. It can be shown by induction that each iteration of the Bellman operator has the form:

$$B^n w(x) = \begin{cases} x_n, & x \leq x_n \\ x, & x > x_n \end{cases}$$

where $x_n = \frac{\delta(1+x_{n-1}^2)}{2} < 1$.

The first application of the Bellman operator to the function w yields:

$$Bw(x) = \max\{x, \delta\} = \begin{cases} \delta, & x \leq \delta \\ x, & x > \delta \end{cases}$$

confirming the claim for $n = 1$ with $x_1 = \delta < 1$. If the claim is true for some $n \geq 1$ (which implies in particular that $x_n < 1$), then we obtain the following for $n + 1$:

$$\begin{aligned} B^{n+1}w(x) &= \max\{x, \delta E[(B^n w)(x)]\} = \max\left\{x, \delta \left(\int_0^{x_n} x_n dz + \int_{x_n}^1 z dz \right)\right\} \\ &= \max\left\{x, \delta \frac{1+x_n^2}{2}\right\} = \begin{cases} x_{n+1}, & x \leq x_{n+1} \\ x, & x > x_{n+1} \end{cases}, \text{ where } x_{n+1} = \frac{\delta(1+x_n^2)}{2} < 1. \end{aligned}$$

Thus, each iteration of the Bellman operator has the form claimed above. Because x_n is a sufficient statistic for $B^n w$, the convergence of $B^n w$ to the fixed point

$$\lim_{n \rightarrow \infty} B^n w(x) = v(x) = \begin{cases} x^*, & x \leq x^* \\ x, & x > x^* \end{cases}$$

is equivalent to the convergence of x_n to the cutoff:

$$x^* = \delta \frac{1+(x^*)^2}{2} < 1.$$

Solving for x^* in the expression above yields:

$$(x^*)^2 - 2^*/\delta + 1 = 0 \Rightarrow x^* = \delta^{-1} \left(1 - \sqrt{1 - \delta^2} \right)$$

since this is the only root of the quadratic that is less than one.¹

¹What if our initial guess was $w(x) = K > 1$ for all $x \in [0, 1]$? Note that this implies $x_0 = K > 1$. The same steps as in the exercise above can be used to show that the threshold at each iteration has the form $x_{n+1} = \delta x_n$ if $x_n > 1$ and $x_{n+1} = \frac{\delta(1+x_n^2)}{2}$ if $x_n < 1$. Clearly, $x_n < 1$ for n large enough, so that the iteration converges to x^* given in part d, confirming once again the prediction of the contraction mapping theorem that the iteration converges to the fixed point of B for any initial guess $w(x)$.

3 Optimal Investment

Note: this problem is in the upper end of how confusing problems will be. The exam should be less (nonstrictly, but well try) confusing than this.

- a. The Bellman equation for the problem is:

$$v(c) = -l + \max\{-c + pEv(c_{+1}), Ev(c_{+1})\}.$$

The following formulation of the Bellman equation is also valid (and equivalent):

$$v(c) = \min\{l + c + pEv(c_{+1}), l + Ev(c_{+1})\}$$

In the utility maximization problems that we have studied, discounting is usually needed to ensure a finite value function and policies that are “interior”. In this problem, we need $l > 0$ (with or without discounting) to ensure that the threshold is strictly positive, $c^* > 0$; and we need discounting and/or a large enough p (and small enough l) to ensure that $c^* < 1$.

Note that in this problem discounting would imply an additional incentive for the agent to *postpone* paying the completion cost. But (for l small enough and p large enough) the agent has an incentive to postpone paying the completion cost for high enough cost draws even without discounting, since cost draws are iid so that it is always possible to get a smaller cost draw in the future.

- b. Adopting the former formulation with the max operator for the Bellman equation (implying that $v(c)$ for any $c \in [0, 1]$ is a nonpositive number) and noting that the optimal policy is defined by a cutoff c^* such that the project is attempted iff $c \leq c^*$, the value function for the problem is given by:

$$v(c) = \begin{cases} -l - c + pEv(c_{+1}), & c \leq c^* \\ -l + Ev(c_{+1}), & c > c^* \end{cases} \quad (1)$$

Note that $Ev(c_{+1})$ is a constant w.r.t. c since c is i.i.d. Because the distribution of c is continuous, it's possible to get a draw of exactly $c = c^*$. At this cutoff, the agent must be indifferent between the two choices:

$$-c^* = (1 - p)Ev(c_{+1}) \quad (2)$$

Taking expectations over c of the value function:

$$Ev(c) = -l + pEv(c_{+1}) - \int_0^{c^*} cdc + \int_{c^*}^1 (1 - p)Ev(c_{+1})dc$$

Applying stationarity ($Ev(c) = Ev(c_{+1})$) and using $(1 - p)Ev(c_{+1}) = c^*$:

$$-c^* = -l - \frac{1}{2}(c^*)^2 - c^*(1 - c^*)$$

$$c^* = \sqrt{2l}$$

The threshold c^* does not depend on the probability p of failing to complete the project. A heuristic explanation is that an increase in the probability of failure p decreases the probability $1 - p$ of success of an attempt today (thus lowering the attractiveness of attempting today) but also increases the expected number of attempts needed for the project to be completed,

which equals $1/(1-p)$ (thus increasing the attractiveness of attempting today in order to avoid additional late fees). Because under no discounting the costs (completion costs and late fees) associated with any trial (now or in the future) are weighted equally, the increase in p leaves the overall attractiveness of attempting the project today unchanged (for any given realization c_t).

Note: what follows is *way* more detailed than what's necessary. Student discretion is advised.

We can also develop some intuition for this result by looking at the sequential formulation of the problem. Given any arbitrary threshold value \tilde{c} , the expected value of the problem is the sum of the total discounted expected project costs paid and the total discounted expected late fees paid. A higher threshold \tilde{c} increases the total expected project costs while decreasing the total expected late fees. At the optimal threshold, the marginal effect of changing the threshold on each of these two components must cancel out. That is, if we're at the optimal threshold c^* and we consider a small perturbation $c^* + \varepsilon$, the marginal reduction in the late fees component must equal the marginal increase in the project costs component. Then, c^* would only depend on $1-p$ if at the optimal c^* , the probability $1-p$ affected these two marginal effects differently. However, in the no discounting case, a decrease in $(1-p)$ affects the marginal impact of c^* on both of these terms by the same proportion, effectively leaving the optimization problem unchanged.

Formally, one can show that, given a candidate threshold \tilde{c} , the present discounted value of expected completion costs and of expected late fees (from the perspective of an agent who hasn't yet observed the current period's draw of c) equals

$$\begin{aligned} ECC(\tilde{c}) &= \tilde{c} \frac{\tilde{c}}{2} + (1 - \tilde{c} + \tilde{c}p)\delta \tilde{c} \frac{\tilde{c}}{2} + \dots \\ &= \frac{\tilde{c}^2}{2[1 - \delta(1 - (1-p)\tilde{c})]} \end{aligned}$$

and

$$\begin{aligned} ELF(\tilde{c}) &= (1 - \tilde{c} + \tilde{c}p)\delta l + \dots \\ &= \frac{l\delta(1 - (1-p)\tilde{c})}{1 - \delta(1 - (1-p)\tilde{c})} \end{aligned}$$

respectively.² The value attained from policy \tilde{c} would then be the negative of the sum of ECC and ELF .

The optimal threshold satisfies

$$\begin{aligned} \frac{dv(c^*)}{dc^*} &= 0 \\ \Leftrightarrow -\frac{\partial ECC}{\partial c^*} - \frac{\partial ELF}{\partial c^*} &= 0 \end{aligned} \quad (3)$$

We can calculate the partials as

$$\frac{\partial ECC}{\partial c^*} = \frac{2c^*(1-\delta) + \delta(1-p)c^*}{2[1 - \delta(1 - (1-p)c^*)]^2}$$

²Note that the agent pays the late fee for the current period regardless of whether he attempts to complete the project.

and

$$\frac{\partial ELF}{\partial c^*} = \frac{\delta l(1-p)}{[1 - \delta(1 - (1-p)c^*)]^2}$$

p in general affects these marginal effects differently, but in the special case of $\delta = 1$ we have

$$\frac{\partial ECC}{\partial c^*} = \frac{1}{2(1-p)}$$

and

$$\frac{\partial ELF}{\partial c^*} = -\frac{l}{c^{*2}(1-p)}$$

so that p indeed affects the two marginal effects by the same proportion and thus does not affect the choice c^* determined by (3). (Note that equation (3) also yields $c^* = \sqrt{2l}$.)

Intuitively, even though the sum of expected completion costs “starts today” and the sum of expected late fees “starts tomorrow”, there is no asymmetry in the impact of p on the two infinite (but convergent) sums since all periods are weighted equally.

c. With a discount factor $\delta \in (0, 1)$ and $p = 0$, the Bellman equation for the problem is:

$$v(c) = -l + \max\{-c, \delta Ev(c_{+1})\}.$$

A threshold rule would then give a value function of:

$$v(c) = \begin{cases} -l - c, & c \leq c^* \\ -l + \delta Ev(c_{+1}), & c > c^* \end{cases}$$

As above, we can obtain an indifference condition for the cutoff c^* :

$$-c^* = \delta Ev(c_{+1})$$

Taking expectations of the Bellman Equation and applying stationarity:

$$-\frac{1}{\delta}c^* = -l - \int_0^{c^*} cdc - \int_{c^*}^1 c^*dc$$

Or:

$$-c^* = -\delta \left(c^* - \frac{(c^*)^2}{2} + l \right)$$

Rearranging gives a quadratic expression:

$$\delta(c^*)^2 + 2(1-\delta)c^* - 2\delta l = 0$$

$$c^* = \frac{\delta - 1 + \sqrt{(1-\delta)^2 + 2\delta^2 l}}{\delta}$$

Note that the other root of the quadratic is negative and can be discarded.

d. It is also possible to generalize the previous analysis to the case where $p \in (0, 1)$. The Bellman equation for this problem is given by:

$$v(c) = -l + \max\{-c + p\delta Ev(c_{+1}), \delta Ev(c_{+1})\};$$

$$v(c) = -l + p\delta Ev(c_{+1}) + \max\{-c, (1-p)\delta Ev(c_{+1})\};$$

The cutoff c^* is defined by the equation:

$$-c^* = (1-p)\delta Ev(c_{+1})$$

Taking expectations of the Bellman equation and substituting:

$$(1-p\delta)Ev(c) = -l - \int_0^{c^*} cdc + \int_{c^*}^1 (1-p)\delta Ev(c_{+1})$$

$$-\frac{1-p\delta}{(1-p)\delta}c^* = -l - \int_0^{c^*} cdc - \int_{c^*}^1 c^*$$

$$-\frac{1-p\delta}{(1-p)\delta}c^* = -l - \frac{1}{2}(c^*)^2 - (1-c^*)c^*$$

And rearranging gives the quadratic polynomial:

$$\delta(c^*)^2 + 2\frac{1-\delta}{1-p}c^* - 2\delta l = 0$$

Whose relevant root is the positive one:

$$c^* = \frac{1}{\delta} \left[\sqrt{\left(\frac{1-\delta}{1-p}\right)^2 + 2\delta^2 l} - \frac{1-\delta}{1-p} \right]$$

Differentiating the quadratic equation totally with respect to p we get

$$\frac{\partial c^*}{\partial p} = -\frac{\frac{1-\delta}{(1-p)^2}c^*}{\delta c^* + \frac{1-\delta}{1-p}}$$

$$< 0$$

so that c^* is decreasing in p .

In terms of the heuristic explanation in part b., now costs incurred in the future are down-weighted due to discounting, so the increase in the expected number of attempts needed for the project to be successfully completed has a smaller (positive) impact on the attractiveness of attempting today (for a given c_t) compared to the (negative) impact of the lower probability of success on the attractiveness of attempting today.

With respect to the second explanation in part b. based on the sequential formulation of the problem, an increase in p has a greater impact through the present discounted value of expected completion costs than through the present discounted value of late fees. This is because the agent pays the late fee for the current period whether or not he attempts to complete the project today, so that he trades off expected completion costs *starting today* and expected late fees *starting tomorrow*. Therefore, because the sum of expected completion costs depends positively on the level of the threshold and the sum of expected late fees depends negatively on the level of the threshold, an increase in the failure rate p lowers the threshold c^* below which the project is attempted (this follows from (3)).

- e. Conditional on the project remaining uncompleted at the start of a period, the probability of completing the project in that period is $c^*(1 - p)$, since:

$$\begin{aligned}\Pr(\text{Project completed}) &= \Pr(\text{Project undertaken}) \cdot \Pr(\text{Project completed} \mid \text{project undertaken}) \\ &= \Pr(c \leq c^*) \cdot (1 - p) \\ &= c^*(1 - p)\end{aligned}$$

Thus, the time T it will take for the project to be completed is a geometric random variable with probability mass function:

$$\Pr(T = t) = [1 - c^*(1 - p)]^t c^*(1 - p),$$

and expectation:

$$\begin{aligned}E(T) &= c^*(1 - p) \sum_{t=0}^{\infty} [1 - c^*(1 - p)]^t \cdot t \\ &= c^*(1 - p) \frac{1 - c^*(1 - p)}{[c^*(1 - p)]^2} \\ &= \frac{1 - c^*(1 - p)}{c^*(1 - p)}\end{aligned}$$

where the second line follows from the fact that, for $\beta < 1$,

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t &= \frac{1}{1 - \beta} \\ \xRightarrow{\text{diff wrt } \beta} \sum_{t=0}^{\infty} \beta^t t &= \frac{\beta}{(1 - \beta)^2}\end{aligned}$$

4 Coding Problem

Please see the solution code file `ps2_julia.html` for sample solution code. We present three figures.

Figure 1 shows the value function $V_0(x)$. Note that the value function is sharply decreasing local to $x = 0$, reflecting the curvature of log utility.

Figure 2 shows the savings policy function $s(x)$. Note that because we have a borrowing limit $s \geq x$ but the minimum income level is $\tilde{y} \geq \frac{1}{10}$, when the consumer's wealth falls low enough, she stops saving entirely. Her savings rate increases sharply local to the boundary where $s = 0$, reflecting a strong desire to save away from the borrowing limit. This reflects the stochastic income process: when the consumer repeatedly hits $\tilde{y} = \frac{1}{10}$, she finds it optimal to decumulate assets with a low savings level. As she moves away from the borrowing limit, her savings rate increases rapidly, reflecting an insurance motive against the idiosyncratic risk of her income process.

To make this clearer, Figure 3 shows the savings policy out of wealth, $s(x)/x$. Note that the savings rate is monotonically increasing and concave (above the borrowing limit) in x as well, reflecting exactly this insurance motive.

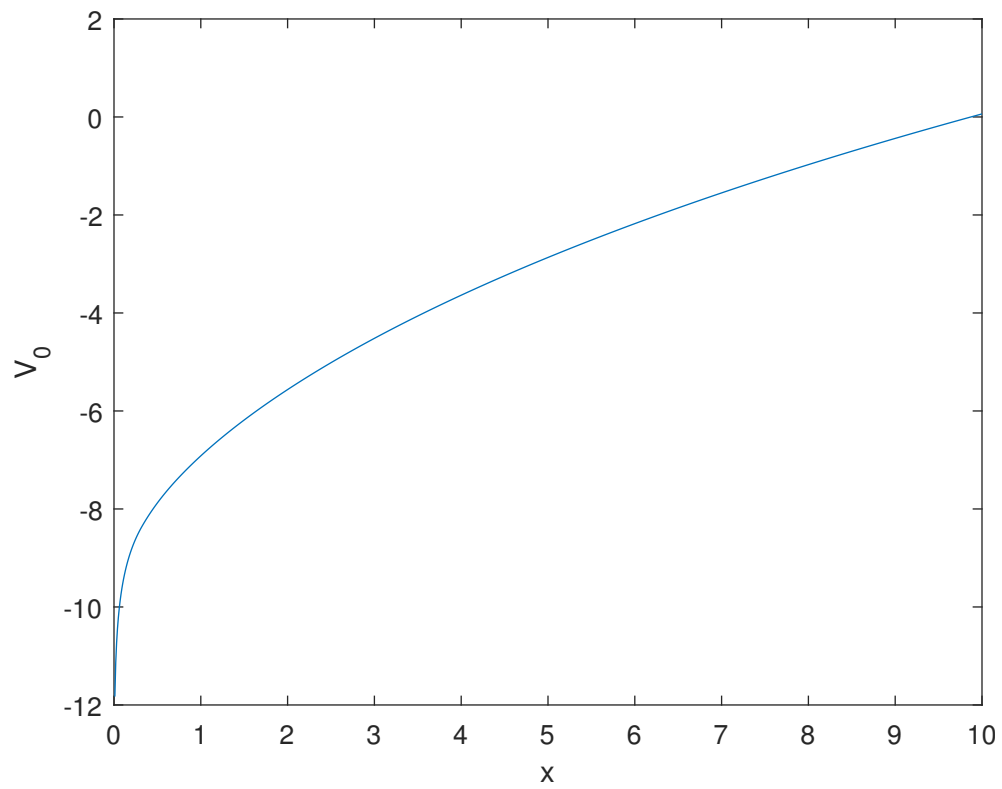


Figure 1: Value function $V(x)$

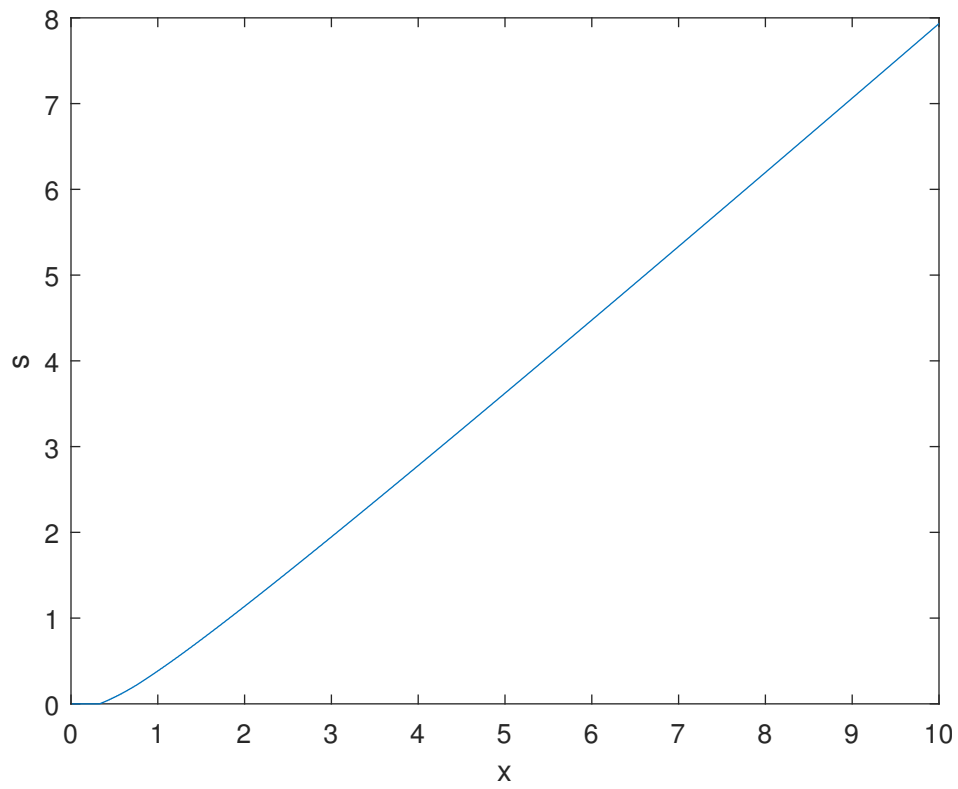


Figure 2: Savings policy $s(x)$

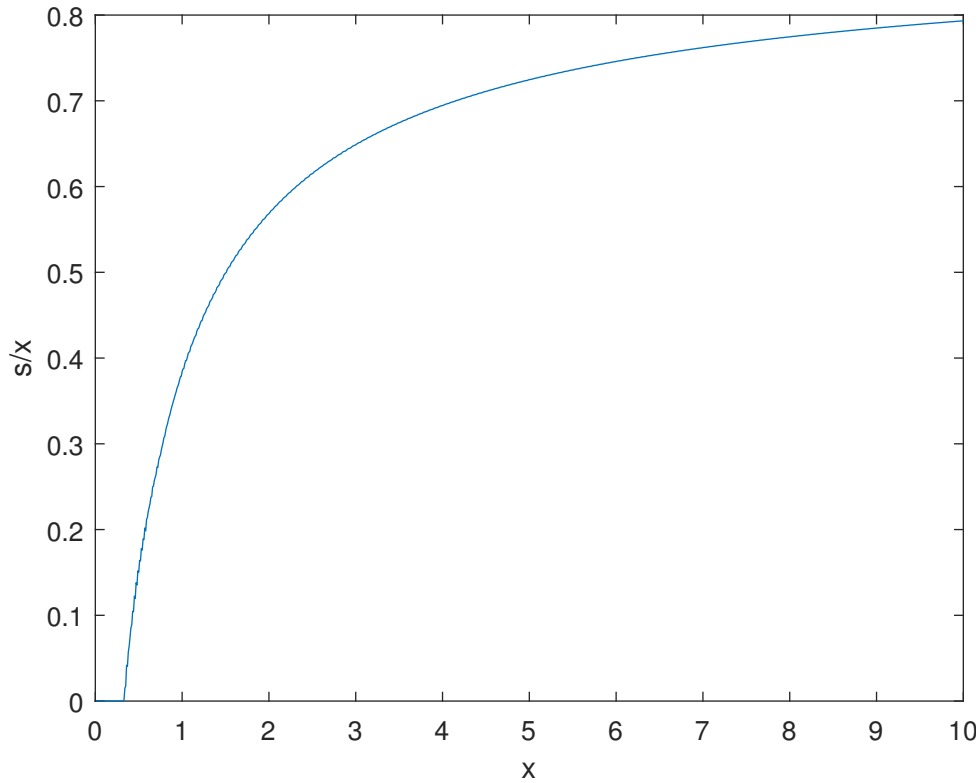


Figure 3: Savings rate $s(x)/x$

5 Optional Problem: The DMP Employment Model

5.1 The Filled Job Problem

The Bellman equation for the employer's value from a filled job is:

$$J(p_t, w) = p_t - w + (1 - s)\delta E[J(p_{t+1}, w)]$$

where max or min does not appear since there is no choice associated with this problem.

The derivative of the flow payoff function $p_t - w$ with respect to w is -1 . Therefore, after accounting for discounting and the probability of separation in a given period, the present discounted value of marginally higher w is $-1(1 + (1 - s)\delta + (1 - s)^2\delta^2 + \dots)$, which yields

$$\frac{\partial J(p_t, w)}{\partial w} = -\frac{1}{1 - (1 - s)\delta}$$

5.2 The Employed Worker's Problem

The Bellman equation for the employed worker's value function takes a similar form to that for the employer's filled job value function, except that separation now implies a non-zero expected continuation value of $E[U(p_{t+1})]$. That is,

$$W(p_t, w) = w + (1 - s)\delta E[W(p_{t+1}, w)] + s\delta E[U(p_{t+1})]$$

and

$$\frac{\partial W(p_t, w)}{\partial w} = \frac{1}{1 - (1 - s)\delta}$$

5.3 The Matching Function

Under the assumption that all unemployed workers have the same probability of filling a given vacancy, a unemployed worker will find a job in period t with probability

$$\frac{m(u_t, v_t)}{u_t} = \kappa u_t^{\alpha-1} v_t^{1-\alpha} = \kappa \theta^{1-\alpha}$$

Similarly, a given vacancy will be filled in period t with probability

$$\frac{m(u_t, v_t)}{v_t} = \kappa u_t^\alpha v_t^{-\alpha} = \kappa \theta^{-\alpha}$$

5.4 The Unemployed Worker's Problem

The Bellman equation for the unemployed worker's value function is given by

$$U(p_t) = z + \delta \kappa [\theta(p_t)]^{1-\alpha} E[W(p_{t+1}, w(p_t))] + \delta (1 - \kappa [\theta(p_t)]^{1-\alpha}) E[U(p_{t+1})]$$

5.5 The Surplus Bellman Equation

The FOC for the Nash bargaining solution is

$$\beta \left(\frac{J(p_t, w)}{W(p_t, w) - U(p_t)} \right)^{1-\beta} \frac{\partial J(p_t, w)}{\partial w} + (1 - \beta) \left(\frac{J(p_t, w)}{W(p_t, w) - U(p_t)} \right)^{-\beta} \frac{\partial W(p_t, w)}{\partial w} = 0$$

Using the derivatives from parts 1 and 2 and multiplying through,

$$-\beta J(p, w) + (1 - \beta)(W(p_t, w) - U(p_t)) = 0$$

$$-\beta J(p, w) + (1 - \beta)(V(p_t) - J(p_t, w)) = 0$$

$$V(p_t) = \frac{J(p_t, w)}{1 - \beta} = \frac{W(p_t, w) - U(p_t)}{\beta} \quad (4)$$

where the second equality uses the definition of $V(p_t)$:

$$V(p_t) = J(p_t, w(p_t)) + W(p_t, w(p_t)) - U(p_t) \quad (5)$$

Substituting the three Bellman equations in (5),

$$\begin{aligned} V(p_t) &= p_t - z - \delta \kappa \theta_t^{1-\alpha} E[(W(p_{t+1}, \hat{w}(p_t)) - U(p_{t+1}))] - \delta E[U(p_{t+1})] \\ &\quad + (1 - s) \delta E[W(p_{t+1}, w)] + s \delta E[U(p_{t+1})] + (1 - s) \delta E[J(p_{t+1}, w)] \\ &= p_t - z - \delta \kappa \theta_t^{1-\alpha} E[(W(p_{t+1}, \hat{w}(p_t)) - U(p_{t+1}))] + (1 - s) \delta E[V(p_{t+1})] \\ &= p_t - z - \delta (\beta \kappa \theta(p_t)^{1-\alpha} - 1 + s) E[V(p_{t+1})] \end{aligned} \quad (6)$$

where the second equality again uses (5) and the third equality uses (4).

The wage does not appear explicitly in the Bellman equation for the surplus from a new match V because we have substituted it out using $w = w(p_t)$. Also note that the details of wage setting do affect V . In particular, the parameter β that regulates how the surplus is allocated between the employer and the employee does enter the Bellman equation for V .

5.6 The Free Entry Condition

The free-entry assumption implies that

$$\begin{aligned} c &= \left(\frac{m(u_t, v_t)}{v_t} \right) J(p_t, w(p_t)) \\ &= \kappa \theta(p_t)^{-\alpha} (1 - \beta) V(p_t) \end{aligned}$$

which implies

$$\begin{aligned} \frac{c\theta_t^\alpha}{\kappa(1-\beta)} &= V(p_t) \\ &= p_t - z - \frac{c\delta(\beta\kappa\theta_t^{1-\alpha} - 1 + s)}{\kappa(1-\beta)} E[\theta_{t+1}^\alpha] \end{aligned} \quad (7)$$

where the second equality uses (6) and the free-entry condition at time $t + 1$.

5.7 Comparative Statics

Simplifying equation (7) using the assumption $E[\theta(p_{t+1})^\alpha] = \theta(p_t)^\alpha$,

$$c - \delta c(1 - s) = \kappa(1 - \beta)(p_t - z)\theta_t^{-\alpha} - c\delta\beta\kappa\theta_t^{1-\alpha} \quad (8)$$

Evaluating (8) at $\theta_t = 1$,

$$\kappa(1 - \beta)(p_t - z) = c - \delta c(1 - s) + c\delta\beta\kappa \quad (9)$$

Totally differentiating equation (8) with respect to $p_t - z$,

$$\begin{aligned} \kappa(1 - \beta)\theta_t^{-\alpha} - \alpha\kappa(1 - \beta)(p_t - z)\theta_t^{-(1+\alpha)} \frac{d\theta_t}{d(p_t - z)} - (1 - \alpha)c\delta\beta\kappa\theta_t^{-\alpha} \frac{d\theta_t}{d(p_t - z)} &= 0 \\ \frac{d\theta_t}{d(p_t - z)} (\alpha\kappa(1 - \beta)(p_t - z)\theta_t^{-(1+\alpha)} + (1 - \alpha)c\delta\beta\kappa\theta_t^{-\alpha}) &= \kappa(1 - \beta)\theta_t^{-\alpha} \end{aligned}$$

Using (9), the elasticity of the vacancy-unemployment ratio at $\theta_t = 1$ is given by

$$\begin{aligned} \frac{d\theta_t}{dp_t} (\alpha c - \alpha \delta c(1 - s) + c\delta\beta\kappa) &= \kappa(1 - \beta) \\ (p_t - z) \frac{d\theta_t}{d(p_t - z)} &= \frac{1 - \delta(1 - s) + \delta\beta\kappa}{\alpha - \alpha\delta(1 - s) + \delta\beta\kappa} \end{aligned} \quad (10)$$

5.8 Calibration

Plugging the given values in (10), we find that

$$(p_t - z) \frac{d\theta_t}{d(p_t - z)} \Big|_{\theta_t=1} = \frac{d \log(\theta_t)}{d \log(p_t - z)} \Big|_{\log(\theta_t)=0} \approx 1.03 \quad (11)$$

A loglinear approximation of $\theta_t = \theta(p_t - z)$ around the value of net labor productivity corresponding to $\theta_t = 1$ is

$$\log(\theta_t) \approx \frac{d \log \theta_t}{d \log(p_t - z)} \Big|_{\log(\theta_t)=0} (\log(p_t - z) - \text{const.}) \quad (12)$$

$$\approx 1.03 \log(p_t - z) + \text{const.} \quad (13)$$

so that the model predicts that

$$\sigma(\log \theta_t) \approx 1.03 \sigma(\log(p_t - z)) \quad (14)$$

Thus, the ratio is predicted to be an order of magnitude lower than the empirical estimates, that is, the (benchmark) DMP model cannot explain the business-cycle volatility of the vacancy-unemployment ratio as the result of labor productivity shocks associated with the business cycle.