MATH 157: Mathematics in the world

Final paper topics

1 How to approach the final project

There are several ways to approach writing your final paper. If you found a problem in class that is of particular interest to you, that is a great start. If you haven't, ask yourself what was the topic that seemed both exciting and approachable. Alternatively, you can also look through the list of topics below.

Once you have a topic or two in mind, I would start by doing two things. First, try to think of questions within that field that you would like to explore. In many cases the questions you ask yourself will not be well-defined or are simply too difficult, but it is still helpful to have a guiding interest in a subject as you approach it. Second, you should look for references. For good or bad, mathematics is a very abstract subject and it is easy to find sources which are virtually unreadable. Your goal should not be to find the most advanced or extensive reference on a given subject, but rather one that spurs your interest and you can see yourself reading.

Last but not least, you should read some part of the sources you found, and write about what you learned during the process. Imagine you are writing an essay to your past self. What would you like to tell yourself? How would you present it? Write an account you wish existed before you started working on your final project.

2 Logic and miscellaneous problems

Tiling problems. See [14].

An interesting question is counting domino tilings over a fixed region. It turns out this is very sensitive to the shape of the region (see the relevant wikipedia article).

Chess problems. See [42]

Proofs from the book. [1]. Beautiful problems with beautiful solutions. Any of the questions in this book could turn into a great final paper with a little extra research.

General combinatorics. Combinatorics is the area of mathematics concerned with finite objects. Among other things, the subject encompasses questions about counting, deciding when certain objects exist, and finding optimal objects.

[7] is a great introduction to the subject. I particularly like the book since it also touches on many of the computational aspects in combinatorics. Other great references are [40] and [23]. The second one is a particularly easy read.

Algebraic combinatorics. The study of polynomials has so many applications to combinatorics, that there is an entire sub-field called *algebraic combinatorics*. This has been a particularly active areas of research during the last three decades. See [36] and [13].

Sperner's lemma and applications. See [1, Chapter 21]. This is a beautiful combinatorial result with applications to many branches of mathematics.

Generating functions. This is a powerful technique for encoding combinatorial problems in terms of infinite power series. It is then possible to use results from calculus and algebra to solve the original combinatorial problems. See [43].

Fractals and dynamics. You have already encountered this topic in a few of the homework exercises. There is an amazing book on the subject by David Mumford [20]. Another good book is [27].

3 Number theory

Quadratic reciprocity. The theory of Quadratic reciprocity, due to Gauss, governs when the equation $x^2 \equiv a \pmod{p}$ has solutions. This is a very classical topic in number theory, and should be very accessible. See [30, Chapter 11], [3, Chapter 9], and [32, Chapter I].

Arithmetic functions. A number of number theoretic functions such as the number of divisors, sum of divisors, and other have interesting properties, summation formulas, and asymptotics. See [30, Chapter 7], [3, Chapters 2, 3], and [22].

Sums of two squares, sums of four squares. Fix integers $n, k \geq 1$, and let $r_k(n)$ denote the number of ways of expressing n as a sum of k perfect squares. It turns out there are closed formulas for $r_2(n)$ and $r_4(n)$ in terms of the divisors of n. See [38, Chapter 10.3]. (This is a great question if you have some background in complex analysis.)

Dirichlet's Theorem on primes in arithmetic progressions. Given coprime integers a and b, Dirichlet proves there are infinitely many primes in the arithmetic progression $\{a + bn \mid n \in \mathbb{Z}\}$. See [39, Chapter 8], [32, Chapter VI], and [3, Chapter 7].

The prime number theorem. The prime-counting function π is asymptotic to $n/\ln n$. See [38, Chapter 7] and [3, Chapter 13].

Analytic number theory. There are a number of other interesting questions in analytic number theory. See [3] and [32].

Cryptography. There are many encryption, key-exchange, and digital signature algorithms worth learning and writing about. Examples include RSA, the Diffie-Hellman key exchange. See [12], [30, Chapter 8], and [16].

Elliptic curves. The study of elliptic curves is one of the most exciting branches of modern number theory. One of the major applications of the subject is cryptography. See [35], or [12] and [16] for cryptographic applications.

p-adic numbers. A p-adic number is an infinite sum $a = \sum_{i=0}^{\infty} a_i p^i$ where $a_i \in \{0, \ldots, p-1\}$. They have a variety of applications in modern number theory. See [17] and [32, Chapter II]. (This could be a good topic if you are interested in learning more about algebra.)

Elementary methods in number theory. [22]. This is a great second book in number theory. Elementary here refers to the fact no proofs use complex analysis, abstract algebra, or other advanced mathematics. The results themselves can be quite fascinating and deep.

Irrational and transcendental numbers. There are many interesting questions about the irrationality or transcendence of well-known constants such as π and e. While seemingly unrelated, these problems are related to a lot of interesting number theory. A great and very readable source on the topic is [24]. To get started, you can also look at [1, Some irrational numbers].

4 Probability

Game theory. See [26] and [2]. For a more combinatorial approach, see [9].

Card shuffling. There is a great paper analyzing whether standard card shuffling techniques (such as the dovetail) produce random results – [5]. In general, Persi Diaconis has a number of other papers which could serve as great final paper topics.

Information theory. See [10] and [15].

Markov chains. See [29] and [25].

Black-Scholes. See [33], [34], [8], and [37].

Other topics in mathematical finance. See [33], [34], [8], and [37].

Stochastic calculus. See [34].

5 Algorithms

Discrete Fourier Transform (DFT) and applications. See [39, Chapter 7] and [21, Chapter 1] for the general theory of discrete Fourier transforms. These techniques have numerous applications to everyday life. For example, the fastest known algorithm for number multiplication uses DFT. Many image compression (e.g., JPEG2000) also use related techniques to aid compression.

JPEG. See [4] and [6].

Wavelets. A great book on the subject is [21]. In addition, there are several introductory articles such as [6], [41], and [18].

Error-correcting codes. See [28] and [19]. For more about information theory, see [10].

Compression. See [31]. For more about information theory, see [15] and [10].

Algorithms in 3D graphics. See [11]. If you find a particular algorithm which interests you, there are many other specific references.

Perlin noise is a great example of simple mathematical idea which found profound applications in graphics. Ken Perlin developed the technique for Disney's Tron (1982). He later won the Academy Award for Technical Achievement (1997).

References

[1] Martin Aigner and Günter M. Ziegler, *Proofs from The Book*, fifth ed., Springer-Verlag, Berlin, 2014, Including illustrations by Karl H. Hofmann. MR 3288091

[2] M. Albert, R. Nowakowski, and D. Wolfe, *Lessons in play: An introduction to combinatorial game theory*, Ak Peters Series, Taylor & Francis, 2007.

[3] Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976, Undergraduate Texts in Mathematics. MR 0434929 (55 #7892)

[4] David Austin, *What is... JPEG?*, Notices Amer. Math. Soc. **55** (2008), no. 2, 226–229. MR 2375024 (2008m:94003)

[5] Dave Bayer and Persi Diaconis, *Trailing the dovetail shuffle to its lair*, Ann. Appl. Probab. **2** (1992), no. 2, 294–313. MR 1161056 (93d:60014)

[6] Catherine Bénéteau and Patrick J. Van Fleet, *Discrete wavelet transformations and undergraduate education*, Notices Amer. Math. Soc. **58** (2011), no. 5, 656–666. MR 2808260 (2012f:42066)

[7] Peter J. Cameron, *Combinatorics: topics, techniques, algorithms*, Cambridge University Press, Cambridge, 1994. MR 1311922 (95j:05002)

[8] Marek Capiński and Tomasz Zastawniak, *Mathematics for finance*, second ed., Springer Undergraduate Mathematics Series, Springer, London, 2011, An introduction to financial engineering. MR 2761467 (2012e:91002)

[9] J. H. Conway, *On numbers and games*, second ed., A K Peters, Ltd., Natick, MA, 2001. MR 1803095 (2001j:00008)

[10] Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, second ed., Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2006. MR 2239987 (2007h:00002)

[11] J.D. Foley, *Computer graphics: Principles and practice*, Addison-Wesley systems programming series, Addison-Wesley, 1996.

[12] P.B. Garrett, *Making, breaking codes: An introduction to cryptography*, Featured Titles for Cryptography Series, Prentice Hall, 2001.

[13] C. D. Godsil, *Algebraic combinatorics*, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993. MR 1220704 (94e:05002)

[14] Solomon W. Golomb, *Polyominoes*, second ed., Princeton University Press, Princeton, NJ, 1994, Puzzles, patterns, problems, and packings, With diagrams by Warren Lushbaugh, With an appendix by Andy Liu. MR 1291821 (95k:00006)

[15] Darrel Hankerson, Greg A. Harris, and Peter D. Johnson, Jr., *Introduction to information theory and data compression*, second ed., Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2003. MR 1990083 (2004i:94027)

[16] Jeffrey Hoffstein, Jill Pipher, and Joseph H. Silverman, *An introduction to mathematical cryptography*, Undergraduate Texts in Mathematics, Springer, New York, 2008. MR 2433856 (2009m:94051)

[17] Neal Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, second ed., Graduate Texts in Mathematics, vol. 58, Springer-Verlag, New York, 1984. MR 754003 (86c:11086)

[18] Demetrio Labate, Guido Weiss, and Edward Wilson, *Wavelets*, Notices Amer. Math. Soc. **60** (2013), no. 1, 66–76. MR 3052464

[19] R.H. Morelos-Zaragoza, *The art of error correcting coding*, John Wiley, 2006.

[20] David Mumford, Caroline Series, and David Wright, *Indra's pearls*, Cambridge University Press, New York, 2002, The vision of Felix Klein. MR 1913879 (2003f:00005)

[21] Amir-Homayoon Najmi, *Wavelets*, Johns Hopkins University Press, Baltimore, MD, 2012, A concise guide. MR 2850446

[22] Melvyn B. Nathanson, *Elementary methods in number theory*, Graduate Texts in Mathematics, vol. 195, Springer-Verlag, New York, 2000. MR 1732941 (2001j:11001)

[23] I. Niven, *Mathematics of choice: Or, how to count without counting*, Anneli Lax New Mathematical Library, Random House, 1965.

[24] Ivan Niven, *Numbers: rational and irrational*, New Mathematical Library, vol. 1, Random House, New York-Toronto, 1961. MR 0130199 (24 #A66)

[25] J. R. Norris, *Markov chains*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 2, Cambridge University Press, Cambridge, 1998, Reprint of 1997 original. MR 1600720 (99c:60144)

[26] Martin J. Osborne and Ariel Rubinstein, *A course in game theory*, MIT Press, Cambridge, MA, 1994. MR 1301776 (95k:90002)

[27] Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe, *Chaos and fractals*, second ed., Springer-Verlag, New York, 2004, New frontiers of science, With a foreword by Mitchell J. Feigenbaum. MR 2031217 (2004m:37061)

[28] W. Wesley Peterson and E. J. Weldon, Jr., *Error-correcting codes*, second ed., The M.I.T. Press, Cambridge, Mass.-London, 1972. MR 0347444 (49 #12164)

[29] Nicolas Privault, *Understanding Markov chains*, Springer Undergraduate Mathematics Series, Springer Singapore, Singapore, 2013, Examples and applications. MR 3100135

[30] Kenneth H. Rosen, *Elementary number theory and its applications*, fourth ed., Addison-Wesley, Reading, MA, 2000. MR 1739433 (2000i:11001)

[31] K. Sayood, *Introduction to data compression*, Morgan Kaufmann series in multimedia information and systems, Morgan Kaufmann Publishers, 2000.

[32] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York-Heidelberg, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7. MR 0344216 (49 #8956)

[33] Steven E. Shreve, *Stochastic calculus for finance. I*, Springer Finance, Springer-Verlag, New York, 2004, The binomial asset pricing model. MR 2049045 (2004m:91003)

[34] ______, *Stochastic calculus for finance. II*, Springer Finance, Springer-Verlag, New York, 2004, Continuous-time models. MR 2057928 (2005c:91001)

[35] Joseph H. Silverman and John Tate, *Rational points on elliptic curves*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992. MR 1171452 (93g:11003)

[36] Richard P. Stanley, *Algebraic combinatorics*, Undergraduate Texts in Mathematics, Springer, New York, 2013, Walks, trees, tableaux, and more. MR 3097651

[37] D. Stefanica, *A primer for the mathematics of financial engineering*, Financial engineering advanced background series, FE Press, 2011.

[38] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, II, Princeton University Press, Princeton, NJ, 2003. MR 1976398 (2004d:30002)

[39] ______, *Fourier analysis*, Princeton Lectures in Analysis, vol. 1, Princeton University Press, Princeton, NJ, 2003, An introduction. MR 1970295 (2004a:42001)

[40] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, second ed., Cambridge University Press, Cambridge, 2001. MR 1871828 (2002i:05001)

[41] James S. Walker, *Fourier analysis and wavelet analysis*, Notices Amer. Math. Soc. **44** (1997), no. 6, 658–670. MR 1452068 (99c:42003)

[42] John J. Watkins, *Across the board: the mathematics of chessboard problems*, Princeton University Press, Princeton, NJ, 2004. MR 2041306 (2004m:00002)

[43] Herbert S. Wilf, *generatingfunctionology*, third ed., A K Peters, Ltd., Wellesley, MA, 2006. MR 2172781 (2006i:05014)