MATH 157: Mathematics in the world
Homework 10 (Due April 23th, 2019 at 1:00PM)

Problem 1

Consider a population of voters uniformly distributed along the ideological spectrum from left ($x = 0$) to right ($x = 1$). Each of the candidates for a single office simultaneously chooses a campaign platform (i.e., a point on the interval $[0, 1]$). The voters observe the candidates' choices, and then each voter votes for the candidate whose platform is closest to the voter's position on the spectrum. If there are two candidates and they choose platforms $x_1 = 0.3$ and $x_2 = 0.6$, for example, then all voters to the left of $x = 0.45$ vote for candidate 1, all those to the right vote for candidate 2, and candidate 2 wins the election with 55 percent of the vote. Suppose that the candidates care only about being elected - they do not really care about their platforms at all!

1. What is the strategy space for each candidate?
2. If there are two candidates, what is the pure-strategy Nash equilibrium.[1]
3. If there are three candidates, exhibit a pure-strategy Nash equilibrium.[2]

Problem 2

Suppose a parent and child play the following game. First, the child takes an action $A \in \mathbb{R}$, that produces income for the child, $I_C(A) = 5 - (A - 3)^2$, and income for the parent, $I_P(A) = 5-(A-1)^2$. Second, the parent observes the incomes I_C and I_P and then chooses a bequest, B, to leave to the child. The child's payoff is $U(I_C + B)$; the parent?s is $V(I_P - B)+U(I_C + B)$, where the utility functions $U(x) = \log x$ and $V(x) = \log(4+x)$.

1. What are the strategy spaces for child and for parents?
2. Use the backwards-induction to find the (subgame perfect) Nash equilibrium of this game.

[1]Assume that any candidates who choose the same platform equally split the votes cast for that platform, and that ties among the leading vote-getters are resolved by coin flips.

[2]There are many Nash equilibria, you only need to find one (and prove it is a Nash equilibrium)!

3. Prove the 'Rotten Kid' Theorem[3]: in the backwards-induction outcome, the child chooses the action that maximizes the family's aggregate income, $I_C(A) + I_P(A)$, even though only the parent's payoff exhibits altruism.

4. (Extra credit) Now consider general functions I_C, I_P, U and V. Assume that all functions are differentiable and strictly concave, and U and V are strictly increasing. Assume also that maximizers of the parent's payoff and the child's payoff exist. Show that the Rotten Kid Theorem holds true.

Problem 3

Consider the following first-price sealed-bid auction. Suppose there are two bidders, $i = 1, 2$. The bidders' valuations v_1 and v_2 for a good are independently and uniformly distributed on $[0, 1]$. The bidders have preferences represented by the utility functions $u_i = x^{\alpha_i}$, where $0 < \alpha_i \leq 1$, $i = 1, 2$. Bidders submit their bids b_1 and b_2 simultaneously. The higher bidder wins the good and pays her bidding price, so that he/she gains $u_i(v_i - b_i)$; the other bidder gets and pays nothing. In the case that $b_1 = b_2$, the winner is determined by a flip of a coin. Find a Bayesian Nash equilibrium (b_1, b_2) in which b_i is a linear function of v_i, $i = 1, 2$.

A Double Auction

There are two players: a buyer and a seller. The buyer's valuation for the seller's good is v_b, the seller's is v_s. The valuations are private information and are drawn from certain independent distributions on $[0, 1]$. The seller names an asking price, p_s, and the buyer simultaneously names an offer price, p_b. If $p_b \geq p_s$, then trade occurs at price $p = (p_b + p_s)/2$; if $p_b < p_s$, then no trade occurs.

The buyer's payoff is

$$\pi_b(p_s, p_b, v_b) = \begin{cases} v_b - (p_b + p_s)/2 & p_b \leq p_s \\ 0 & p_b < p_s \end{cases}$$

The seller's payoff is

$$\pi_s(p_s, p_b, v_s) = \begin{cases} (p_b + p_s)/2 - v_s & p_b \leq p_s \\ 0 & p_b < p_s \end{cases}$$

There are many, many Bayesian Nash equilibria of this game. In the following, we will consider two types.

[3] See Rotten Kid Theorem

Problem 4 - One-price equilibria

For any value $x \in [0, 1]$, which is given exogenously and is known by both players, the *one-price strategies at* x are as follows:
The buyer offers x if $v_b \geq x$ and 0 otherwise;
The seller demands x if $v_s \leq x$ and 1 otherwise.

1. Show that if buyer and seller both play the one-price strategies at x is a Bayesian Nash equilibrium.
2. Compute the region where the trade occurs.
3. A trading mechanism is called Pareto efficient if the item is finally given to the agent who values it the most. Use your computation in the previous part to show that the one-price strategies at x is not Pareto efficient. [4]
4. We define the expected total gains from trade to be the sum of expected gains of the buyer and the seller. Compute the expected total gains from trade for one-price equilibrium at x.

Problem 5 - Linear equilibria

Now we look for *linear strategies*

$$p_i(v_i) = a_i + c_i v_i,\ i = s, b,$$

with $a_i \geq 0$ and $c_i > 0$.

1. Given the seller's linear strategy $p_s(v_s) = a_s + c_s v_s$, what is the buyer's expected payoff
$$E_{v_s}[\pi_b(p_s(v_s), p_b, v_b)]?$$
2. To maximize $E_{v_s}\pi_b[p_b, p_s(v_s)|v_b]$, what is the buyer's best response?[5]
3. Similarly, given the buyer's linear strategy $p_b(v_b) = a_b + c_b v_b$, what is the seller's expected payoff
$$E_{v_b}[\pi_s(p_s, p_b(v_b), v_s)]?$$
4. To maximize $E_{v_b}[\pi_s(p_s, p_b(v_b), v_s)]$, what is the seller's best response?
5. Use all the previous part to find the linear Bayesian Nash equilibrium.
6. Compute the region where the trade occurs, and show that the linear Bayesian Nash equilibrium is not Pareto efficient.[6]

[4]This is a special case of Myerson-Satterthwaite theorem
[5]You should be surprised and excited that the expression for buyer's best response is a linear function!
[6]Again, this is a special case of Myerson-Satterthwaite theorem

7. Compute the expected total gains from trade for the linear Bayesian Nash equilibrium. How does it compare to the one-price equilibria?[7]

[7] In fact, Myerson and Satterthwaite (1983) showed that, for the uniform valuation distributions, the linear equilibrium yields higher expected total gains than any other Bayesian Nash equilibria of the double auction (including but far from limited to the one-price equilibria).