MATH 157: Mathematics in the world

Notes 10 (February 28, 2019)

Generating functions

Let us consider the function

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{i=0}^{\infty} x^i$$

We want to consider it as a series; series are very helpful in solving many mathematical problems.

1 - A bit of practice

Try to write the following series as a fraction.

1. $1 + 2x + 4x^2 + 8x^3 + \ldots = \sum_{k=0}^{\infty} 2^k x^k$
2. $x - x^2 + x^3 - x^3 + \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} x^k$
3. $1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{k=0}^{\infty} (k+1) x^k$

4*. $1 + (r+1)x + \binom{r+2}{2} x^2 + \binom{r+3}{3} x^3 + \ldots = \sum_{k=0}^{\infty} \binom{r+k}{k} x^k$ [1]

2 - Another bit of practice

Try to write the following series in a "closed" form.

1. $\sum_{k=0}^{n} x^k$
2. $\sum_{k=0}^{n} \binom{n}{k} x^k$
3. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ [2]
4. $\sum_{k=1}^{\infty} \frac{x^k}{k}$

[1] the number $\binom{n}{k}$, called *binomial coefficient* (we will call it n *choose* k) is equal to $\frac{n!}{k!(n-k)!}$

[2] in general, we assume that $0! = 1$ and that $1! = 1$.

3 - Using generating functions

Consider the following two sequences:

1. $a_0 = 0,\ a_{n+1} = 2a_n + 1,$
2. $b_0 = 0,\ b_{n+1} = 2b_n + n.$

Express the series

$$\sum_{k=0}^{\infty} a_k x^k \quad \sum_{k=0}^{\infty} b_k x^k$$

as fractions, and use it to find a closed expression for a_n and b_n.

4 - Fibonacci, 1st part

Try to write the following series as a fraction.

$$\sum_{k=0}^{\infty} F_k x^k$$

where $F_0 = 0$, $F_1 = 1$ and F_n is the n-th Fibonacci number.

5 - Fibonacci, 2nd part

Use the previous exercise and the following identity

$$\frac{1}{(x-a)(x-b)} = \frac{1}{ab(a-b)}\left(\frac{a}{1-\frac{x}{b}} - \frac{b}{1-\frac{x}{a}}\right)$$

to find a closed formula expression for the Fibonacci numbers.

6 - Another sequence

Suppose we have a sequence C_n such that $C_0 = 1$ and $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$. Follow the same path of the previous two exercises to find a closed formula for C_n.

As a hint recall that:

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 + ... = \sum_{0}^{\infty}(-1)^k \binom{\frac{1}{2}}{k} x^k, \ \textit{where}$$

$$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\dots(\frac{1}{2}-k+1)}{k!} \ \textit{is a generalized binomial coefficient.}$$

7 - Partitions

Let d_n be the number of ways you can write n as a sum $a + 2b + 3c$, where a, b, c are nonnegative integers (let's impose $d_0 = 1$). Find the series

$$\sum_{k=0}^{\infty} d_k x^k$$

8 - A sum

For a given n integer, evaluate the sum

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

Extra

9 - Subsets

Let p be an odd prime number. Find the number of subsets A of the set $\{1, 2, \ldots, 2p\}$ such that

- A has exactly p elements, and
- the sum of all the elements in A is divisible by p.