# MATH 157: Mathematics in the world Notes 11 (March 5, 2019)

## Generating functions

## 1 - Using generating functions

Consider the following two sequences:

- 1.  $a_0 = 0, a_{n+1} = 2a_n + 1,$
- 2.  $b_0 = 0$ ,  $b_{n+1} = 2b_n + n$ .

Express the series

$$\sum_{k=0}^{\infty} a_k x^k \quad \sum_{k=0}^{\infty} b_k x^k$$

as fractions, and use it to find a closed expression for  $a_n$  and  $b_n$ .

## Probability and counting

The probability is defined as

 $\frac{\#\text{positive cases}}{\#\text{all cases}}$ 

if all cases have the same probability to happen. Otherwise, we would have 1/2 of probability of winning the lottery!

#### 1 - Walking in Manhattan

Imagine a city whose streets form a perfect grid. Starting at the point with coordinates (0,0), how many ways are there to walk to the intersection (m,n) in m + n steps? What is the probability of reaching the intersection (m,n) in m + n steps?

### 2 - An even number of heads

You flip n fair coins independently. What is the probability to get an even number of heads?

#### **3** - Comparing n an n+1

Amy has n + 1 fair coins, while her brother Brad only has n. If both flip all of their coins, what is the probability that Amy will end up with more heads than Brad?

## 4 - Flippant juror

As part of his jury duty, an undergraduate student was chosen to participate in a jury of three members (majority rules). To express his disinterest in the case, he decides to vote by flipping a fair coin. The other two members of the jury independently will make the right decision with probability p. How does this arrangement compare to a single member jury which decides on the correct outcome with the same probability p?

## Stirling formula

Stirling formula allows to approximate large expressions that involve factorials. It says that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

where  $\sim$ , as before, means that the ratio of the two quantaties tends to 1 as n tends to infinity.

#### 5 - Even split

What is the probability of getting exactly 50 heads if you flip 100 fair coins? Can you find a concrete figure approximating the result without using additional computational power?

## 6 - Unseen deck ordering

Prove that there exists an ordering of a deck of 36 cards that has never been obtained after a shuffle. Hint: the world population is 7.44 billion people and the age of the universe is roughly 13.8 billion years.

## More probability

### 7 - The second best

Eight players enter a tennis tournament with a bracket structure. Suppose that the best player will always win against the second best, who in turn wins against the next best, and so on. If the first round allocation is random, what is the probability that the second best player will play in the final? What if the tournament included  $2^n$  players?

### 8 - The rock-paper-scissors tournament

Two friends enter a rock-paper-scissors bracket with 6 other players. All players are evenly matched, so the probability of winning any particular game is 1/2. If the initial arrangement is random, what is the chance that the two friends will face each other at some point during the tournament? What is the answer if the tournament has  $2^n$  players in total?