MATH 157: Mathematics in the world

Notes 12 (March 7, 2019)

Probability and Discrete Random Variables

A *Probability Space* consists of

1. A sample space Ω, which is a set of all possible outcomes;
2. A set of events $\mathcal{F}$, where each event is a set containing zero or more outcomes;
3. A function $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$, which assigns each event its probability.

A *Random Variable* X is a function $X : \Omega \longrightarrow \mathbb{Z}$ where for any number a, $X^{-1}(a)$ is an event.
Given a random variable X, we define its *expection* as

$$\mathbb{E}(X) = \sum_k k \cdot \mathbb{P}(X = k).$$

1 - First six

How many times do you need to throw a die on average before you get a 6 (including the last throw)?

2 - Collecting famous mathematicians

Tasty-o-crunch, a new brand of cereal, decided to include cards featuring famous mathematicians with their product. Each box contains one of 10 possible cards all of which are equally likely to appear. How many boxes of cereal do you need to buy on average to get a complete set?

3 - St. Petersburg paradox

A casino offers a game in which a fair coin is tossed at each stage. The initial steak is \$2. If the coin comes up tails, you receive the steak. If we get heads, the steak is doubled, and the process repeats. How much should they charge for this game?

What if every time we get heads, the steak is increased by one? How much should they charge?

Conditional Probability

Given two events $A, B \subset \Omega$, the *conditional probability of A given B* is defined as

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Given a random variable X, it is also possible to define its *conditional expectation* as

$$\mathbb{E}(X \mid B) = \sum_k k \cdot \mathbb{P}(X = k \mid B).$$

1. **Law of total probability.** Suppose $E_1, \ldots, E_n \subset \Omega$ are mutually exclusive events such that $\bigcup_{i=1}^n E_i = \Omega$. Given any event $A \subset \Omega$, we can compute its probability by

   $$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A \mid E_i) \cdot \mathbb{P}(E_i).$$

   Similarly, given a random variable X, we can express its expectation in terms of its conditional expectations:

   $$\mathbb{E}(X) = \sum_{i=1}^n \mathbb{E}(X \mid E_i) \cdot \mathbb{P}(E_i).$$

2. **Multiplication rule of conditional probability.** Given events $E_1, \ldots, E_n \subset \Omega$, the probability of their intersection can be computed using conditional probability as

   $$\mathbb{P}(E_1 \cap \cdots \cap E_n) = \mathbb{P}(E_1)\, \mathbb{P}(E_2 \mid E_1)\, \mathbb{P}(E_3 \mid E_1 \cap E_2) \ \cdots \ \mathbb{P}(E_n \mid E_1 \cap \cdots \cap E_{n-1}).$$

3. **Bayes formula.** Given two events $A, B \subset \Omega$, the ratio of the conditional probabilities both ways can be expressed as

   $$\frac{\mathbb{P}(A \mid B)}{\mathbb{P}(B \mid A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

4 - Conditional first six

How many times do you need to throw a die on average before you get a 6 conditioned on the event that all throws before the first 6 are even numbers (including the last throw)? (Be careful about your argument!)

5 - Two sons

1) Mr. Smith has two children and one of them is a boy. What is the probability both children are boys? Later you learn that his elder child is a boy. What is the probability now?

2) You go to dinner with your guy friend. During the conversation, he mentioned he is one of two children. What is the probability that they are both boys? Later on, he clarified that he is the older sibling. What is the probability now?

6 - Two sixes

You throw a fair die until two consecutive throws come up 6. What is the average number of throws before the game terminates (including the last throw)?