Math 157: Lecture 13
A Probability Potpourri

Today's agenda: The first two problems introduce concepts that could be further explored in your final project. The remaining problems are classics of the genre.

Another dice game

You may roll a die as a many times as you want, according to the following rules. If you roll a six, the game ends and you win nothing. Otherwise, you may stop at any moment and win the sum of all previous rolls.

- What is the optimal strategy for playing this game, supposing you want to maximize its expected value?
- Can you estimate the expected value of the game with optimal play? (Hint: If optimal play is too hard to compute by hand, try computing the value of a simpler, suboptimal strategy to get a lower bound.)
- How much would you pay to play this game a single time?

If this problem interests you, you may want to consider the Optional Stopping Theorem or the St. Petersburg Paradox for your final project.

World Series

In an alternate universe, the Red Sox and Yankees are about to play in the World Series. You want to bet \$10 on the Red Sox winning the entire series, so that you gain \$10 if they win and lose your stake of \$10 if they lose.

However, due to Massachusetts gambling laws, you can only bet on individual games, not the final outcome. For each game, you may bet some amount of money on either team, with payouts as above (1 : 1 odds). How should you place your bets in order to replicate your desired bet on the outcome of the series?

- First, assume the series is at most three games long, so that the first team to two wins takes the series. How should you bet?
- Can you generalize your approach to a more realistic best-of-seven series?

Related projects include the binomial asset pricing model and the Black–Scholes–Merton model for financial derivatives.

The Monty Hall problem

You are a contestant on a game show, where you are given the choice of three doors. Behind one door is a car, but behind the other two are goats. These "prizes" were placed randomly by the host. You pick door 1. The host then examines the remaining doors and opens one with a goat. (If both have goats, he picks randomly). Suppose the host opens door 3. He asks you, "Do you want to switch and pick door 2 instead?" Should you switch?

Fair from unfair

You are given a coin, but you worry it is not fair. Is it possible to simulate a fair coin using it? You may assume successive tosses are independent.

If the coin is biased so that it shows heads with probability $p \in (0, 1)$, what is the expected number of coin tosses your procedure will take?

More first-step analysis

You toss a fair coin until two consecutive throws come up heads. What is the expected number of tosses before the game ends?

Geometric probability warmup

The following game is often offered at carnivals. There is a table whose surface is ruled in 1-inch squares. You take a penny from your pocket (3/4-inch in diameter), and after stepping several feet away from the table, you throw the penny onto it. If the penny falls within a single square, you get 10 cents and your penny back. If the penny falls on one of the rulings, you lose your penny. Would you play this game?

Serendipity

After talking about the movie *Serendipity* on a dating website, a potential couple decides to go on a random date. Each person will arrive at a predetermined restaurant randomly between 7 and 8 PM and wait for exactly 15 minutes, but no later than 8 PM. If they overlap, the relationship is meant to be; otherwise, they move on. What is the chance they meet?

Points on a semicircle

If we choose n points randomly on a fixed circle, what is the probability they will lie in a semicircle? (Hint: Try working out small n first.)

Bayesian coins

There are five coins in a box; one has heads on both sides, while the other four are normal. You reach into the box, draw a coin at random, and flip the coin three times. If you get three heads in a row, what is the probability you drew the two-headed coin?

Bayesian medicine

A patient goes to see a doctor. The doctor performs a test with 99 percent reliability—that is, 99 percent of people who are sick test positive and 99 percent of the healthy people test negative. The doctor knows that only 1 percent of the people in the country are sick. If the patient tests positive, what are the chances the patient is sick?

(Taken from an article by Chris Wiggins in Scientific American: `https://www.scientificamerican.com/article/what-is-bayess-theorem-an/`)

Airplane boarding

Spring break has begun, and there are 100 Harvard students at Boston Logan Airport waiting to board a plane. The first student in line realizes that she has lost her boarding pass, so she decides to take a random seat instead. Every student that boards the plane after her will either take their assigned seat, if it is available, or a random seat if the assigned seat is taken.

What is the probability that the last student who boards will end up in their assigned seat?