MATH 157: Mathematics in the world
Notes 16 (March 28, 2019)

1 Continuous variables

If X is a continuous random variable, we will call **probability density function (PDF)** of X the function $f_X(x)$ such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

Another important function is the **cumulative distribution function (CDF)** of X the function $F_X(x)$ such that

$$\mathbb{P}(-\infty < X \leq a) = F_X(x).$$

Notice that this function $F_X(x)$ is increasing, and by the Fundamental Theorem of Calculus we have

$$\frac{dF_X(x)}{dx} = f_X(x).$$

To find the expected value of X, we can use the formula

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)dx.$$

To find the expected value of a random variable $g(X)$, by the Chain Rule we can use the formula

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

The variance of a variable X is the expected value of the square of the difference with the mean value; in formulas

$$Var(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \int_{-\infty}^{\infty} x^2 f_X(x)dx - \left(\int_{-\infty}^{\infty} x f_X(x)dx\right)^2$$

Upright table

A carpenter with a good sense of humor attached three legs to a circular table-top in random uniformly distributed positions. What is the chance the table will stay upright?

Point on a segment

The unit interval is cut in two using a randomly chosen point. What is the expected length of the shorter piece? What about the longer one?

Two points on a segment

The unit interval is cut at two independently chosen random points. What is the expected length of the left piece?

2 Random walks

A bug's life

A small flea Alice lives on an 1-dimensional island of width m. n inches to her left is the Cliff of Doom, which drops to a raging sea filled with flea-eating monsters. $m - n$ inches to her right is the Pit of Disaster, which falls to gap with a stream of molten lava.

Each second, Alice hops 1 inch to the right or the left with equal probability, independent of the direction of all previous hops. Our job is to analyze the life of Alice. Does she have any chance of avoiding a fatal plunge? If not, what is the probability of her falling into the Cliff or Pit respectively? What is her life expectancy?

What happens if Alice hops 1 inch to the right with probability p and to the left with probability $q = 1 - p$?

A bug's new life

A sudden plate movement has changed the geography: the gap of Pit of Disaster has been filled. To Alice's right, there is an endless flat plateau. Does Alice have a chance to survive now? How about her life expectancy now?

Gambler's ruin

This summer, I am planning to go to Las Vegas and try to earn some cash. The game I am going to play is Roulette. To summarize the game, I will win \$1 with probability

$$\frac{18}{38} \approx 0.474,$$

and lose \$1 with probability

$$\frac{20}{38} \approx 0.526.$$

As a mathematician, I am very rational. So I will choose to leave either if I win \$100 or if I lose all my money. To improve my chance of going home a winner, I can bring more money. What will be the probability of wining if I bring

1. \$100
2. \$1,000
3. \$1,000,000,000

Random Walk in Circles

We are going to play a game of passing pomelo. Suppose there are $n+1$ people numbered by $0, 1, ..., n$ standing in a circle. I, being person 0, have a pomelo to start passing around. Every time, the person

who is holding the pomelo has equal probability of passing it to the left or right. The game ends when all but one have touched the pomelo, in which case the final person who never touches it will win the pomelo. Where do you want to stand at the beginning?