MATH 157: Mathematics in the world

Notes (April 2, 2019)

These notes are inspired by [Almgren and Chriss, 1999]. I have tried to preserve the notation while simplifying it where possible. Please consult the original paper if you are interested in exploring the topic further.

1 Trading strategies

Imagine that we hold $X \geq 0$ units of a security, and we would like to unwind them in a controlled fashion over $N \geq 1$ steps. A *trading trajectory* is a tuple $x = (x_0, \ldots, x_N)$ which prescribes how many units we plan to hold at each point in time. We will require

$$X = x_0 \geq x_1 \geq \cdots \geq x_{N-1} \geq x_N = 0.$$

Equivalently, we can also specify the *trade list* $n = (n_1, \ldots, n_N)$, where $n_k = x_{k-1} - x_k$.

A *trading strategy* is a rule for determining n_k from information available after step $k-1$.

2 Price dynamics

Suppose the initial security price is S_0. We model the price evolution similarly to a random walk,

$$S_k = S_{k-1} - \Delta_k,$$

where the random variables $\Delta_k = \Delta_k^{\text{ex}} + \Delta_k^{\text{end}}$ may include both exogenous and endogenous effects.

Usually, it is not possible to execute a trade instruction n_k exactly at the corresponding observable price S_{k-1}. Instead, we assume we can execute at a price $\tilde{S}_k = S_{k-1} - \Delta_k^{\text{inst}}$, where Δ_k^{inst} denotes the instantaneous (temporary) impact of the current single transaction.

It is reasonable to assume that Δ_k^{ex} are IID with mean 0 and variance σ^2 (similar to a random walk). Similarly, we will also assume that Δ_k^{end} and Δ_k^{inst} are deterministic in n_k.

3 Cost

Given the setup above, the *total cost of trading* (also called *implementation shortfall*) of a trajectory x is

$$C(x) = XS_0 - \sum_{k=1}^{N} n_k \widetilde{S}_k = \sum_{k=1}^{N} n_k \Delta_k^{\text{inst}} + \sum_{k=1}^{N} x_k \left(\Delta_k^{\text{ex}} + \Delta_k^{\text{end}} \right).$$

Prior to trading $C(x)$ is a random variable; we will, respectively, use $E(x)$ and $V(x)$ to denote its expectation and variance.

Given the assumptions we made above, we can write

$$E(x) = \sum_{k=1}^{N} n_k \Delta_k^{\text{inst}} + \sum_{k=1}^{N} x_k \Delta_k^{\text{end}},$$
$$V(x) = \sigma^2 \sum_{k=1}^{N} x_k^2.$$

4 Questions

1. Come up with a list of factors which may influence Δ_k^{ex}, Δ_k^{end}, and Δ_k^{inst}.
2. What is a good objective function to determine optimality? How does it combine $E(x)$ and $V(x)$? Are there any choices involved and what do they signify?
3. Suppose $\Delta_k^{\text{end}} = \Delta_k^{\text{inst}} = 0$. What trading strategies would you recommend?
4. What if we only assume $\Delta_k^{\text{end}} = 0$, but we can also ignore $V(x)$?
5. Assume $N = 2$. Can you find closed form solutions for simple expressions of Δ_k^{inst} and Δ_k^{end}? Is there some intuition you can get from these solutions?
6. Can you think of simple forms for Δ_k^{inst} which yield a closed-form solution?
7. In what directions would you like to further develop this topic?

References

[Almgren and Chriss, 1999] Almgren, R. and Chriss, N. (1999). Optimal execution of portfolio transactions. *Journal of Risk*, pages 5–39.