MATH 157: Mathematics in the world

Notes 19 (April 9, 2019)

Game of chance

We are going to play a game. We will choose 3 numbers randomly uniformly from the set of integers in $\{1, 2, ..., 100\}$ and compute their product. If the first digit is $1, 2, 3$, I win. Otherwise, you win. How much do you pay to play the game?

Some other statistics

1. How do you think the leading digit of a country's population are distributed?
2. How about the leading digit of a country's area?
3. How about the leading digit of person's height?
4. How about the leading digit of a license for vehicle?

A special mathematical statement of Benford's law

Consider the sequence $x_i = 2^i$. Let $N_{k,n}$ denote the number of x_is with a leading digit k for $i \le n$, then

$$\lim_{n\to\infty} \frac{N_{k,n}}{n} = \frac{\log(k+1) - \log k}{\log 10}$$

1 Dynamical system - Rigid rotation

Alice, the flee, lives on a one dimensional circle. Without knowing her being trapped in a circle, she tries to escape from reality by running in the same direction with constant speed. Being discrete in nature, our camera captures the position Alice every 1ms. We want to analyze the life of Alice on this circle.

A (discrete) dynamical system is can be mathematical model for our study. We let $S^1 = \mathbb{R}/\mathbb{Z}$. A map T_α of the form

$$\begin{aligned} T_\alpha : S^1 &\longrightarrow S^1 \\ x \mod 1 &\mapsto x + \alpha \mod 1 \end{aligned}$$

is called a *rigid rotation*.

Rational rotation

If Alice is running at a rational speed, what happens to our pictures?

Using our mathematical formulation: if α is a rational number, what happens if you consider the iterations of a point x under T_α, i.e., what happens to the sequence $x, T_\alpha(x), T_\alpha(T_\alpha(x)), \ldots$?

Irrational rotation

What if α is irrational?

Let $f : S^1 \longrightarrow \mathbb{R}$ be a function (for example, it can be a function that tells you the temperature at a particular point $x \in S^1$). We are interested in two different averages of the function f: time average and space average.

Time average

Let $x \in S^1$, the time average starting at x is defined as

$$T(x) = \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} f(T_\alpha^i(x))}{n}$$

Space average

The space average S is defined as

$$S = \int_{S^1} f(x)dx$$

(Unique) Ergodicity Theorem

If α is irrational, then for every $x \in S^1$, $T(x) = S$.

Application to Benford's law

How does this help us to prove the Benford's Law for the sequence $x_i = 2^i$?

Relations to Mandelbrot set

Where can we find rotations on the Mandelbrot set?