MATH 157: Mathematics in the world

Notes 20 (April 11, 2019)

1 Static Games of Complete Information

The Prisoners' Dilemma

Two suspects are arrested and charged with a crime. The police lack sufficient evidence to convict the suspects, unless at least one confesses. The suspects are held in separate cells and told 'If only one of you confesses and testies against your partner, the person who confesses will go free while the person does not confess will surely be convicted and given a 20-year jail sentence. If both of you confess, you will both be convicted and sent to prison for 5 years. Finally, if neither of you confesses, both of you will be convicted of a minor offence and sentenced to 1 year in jail.' What should the suspects do?

The *normal-form* (also called *strategic-form*) representation of an n-player game specifies the players' strategy spaces $S_1, ..., S_n$ and their payoff functions

$$u_i : S_1 \times ... \times S_n \longrightarrow \mathbb{R}.$$

The game is denoted by $G = \{S_1, ..., S_n, u_1, ..., u_n\}$.

We will denote

$$\begin{aligned} S_{-i} &= S_1 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_n \\ S &= S_1 \times ... \times S_n. \end{aligned}$$

We will also use s_{-i} and s to denote an element in S_{-i} and S respectively.

The *best response* for player i to a combination of other players' strategies $s_{-i} \in S_{-i}$, denoted by $R_i(s_{-i})$, is referred to as the set of maximizers of

$$\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

The strategies $(s_1^*, ..., s_n^*) \in S$ are a *Nash Equilibrium* if

$$s_i^* \in R_i(s_{-i}^*) \; \forall i = 1, ..., n.$$

In other words,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \; \forall i = 1, ..., n.$$

Splitting a dollar

Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, s_1 and s_2, where $0 \le s_1, s_2 \le 1$. If $s_1 + s_2 \le 1$, then the players receive the shares they named; if $s_1 + s_2 > 1$, then both players receive zero. What are the (pure-)strategy Nash equilibria of this game?

Cournot Model of Duopoly

If a firm produces q units of a product at a cost of c per unit and can sell it at a price of p per unit, then the firm makes a net profit

$$\pi = pq - cq.$$

The firm can decide p and q to maximize its net profit. Suppose firms 1 and 2 produce the same product. Let q_i be the quantity of the product produced by firm i. Let $Q = q_1 + q_2$, the aggregate quantity of the product. Since firms produce the same product, the

firm which sets the higher price has no market. Thus, they sell the product at the same price, the market clearing price

$$P(Q) = \begin{cases} a - Q & Q \le a \\ 0 & Q > a \end{cases}$$

Let the cost of producing a unit of the product be c, and we assume that $c < a$ and is the same for both firms. How much shall each firm produce?

Matching Pennies - Mixed Strategy

Each player has a penny and must choose whether to display it with heads or tails facing up. If the two pennies match (i.e., both are heads up or both are tails up) then player 2 wins player 1's penny; if the pennies do not match then 1 wins 2's penny. What is a Nash equilibrium of this game?

Theorem 1 (Nash, 1950) *In the n-player normal-form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, if n is finite and S_i is finite for every i, then there exists at least one Nash equilibrium, possibly involving mixed strategies.*

2 Dynamic Games of Complete Information

By complete information, we mean that the payoff functions are common knowledge. We will consider 2 cases:

1. Dynamic games with complete and perfect information.

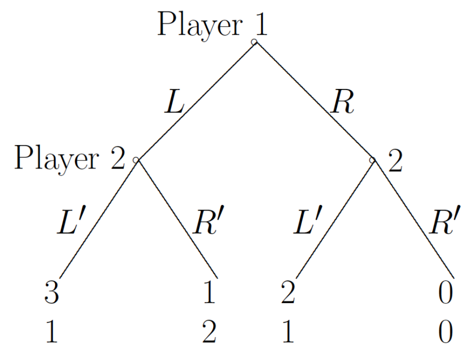
   By perfect information, we mean that at each move in the game, the player with the move knows the full history of the play of the game thus far.

2. Dynamic games with complete but imperfect information

   At some move the player with the move does not know the history of the game.

A first example

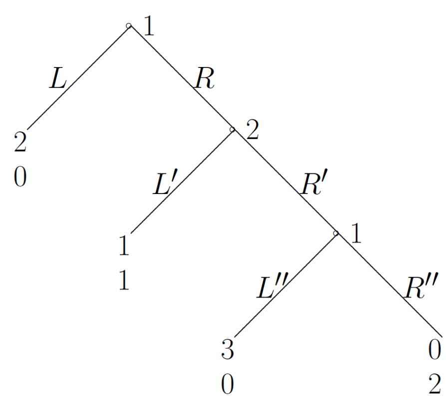
Player 1 chooses an action L or R. Player 2 observes player 1's action and then chooses an action L' or R'. Each path (a combination of two actions) in the following tree followed by two payoffs, the first for player 1 and the second for player 2.



This game tree is usually referred to as the *extensive-form representation* of a game.

A three-move game

Consider the following game tree. The first payoff is for player 1 and the second is for player 2.



Stackelberg Model of Duopoly

Consider a dominant firm moving first and a follower moving second.
1. Firm 1 chooses a quantity $q_1 \geq 0$.
2. Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$.

3. The payoff to firm i is the profit

$$\pi_i = (P(Q) - c)q_i,$$

where $Q = q_1 + q_2$ and

$$P(Q) = \begin{cases} a - Q & Q \leq a \\ 0 & Q > a \end{cases}$$

What does the backward induction give us?

Observation: Firm 2 has more information in Stackelberg game than in Cournot game.

Question: In which game does firm 2 make more proit?