# MATH 157: Mathematics in the world

Notes 21 (April 16, 2019)

## 1 Static Games with incomplete information (Bayesian Games)

The *normal-form representation* of an $n$-player static Bayesian game specifies:

1. Players' type spaces $T_1, ..., T_n$;
2. Players' action spaces $A_1, ..., A_n$; [1]
3. Players' beliefs on other peoples' type $P_1, ..., P_n$; [2]
4. Players' payoff function $u_1, ..., u_n$.

The strategies $s^* = (s_1^*, ..., s_n^*)$ are a *Bayesian Nash equilibrium* if for each player $i$ and for any type $t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} E_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i),$$

where

$$E_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i) = \int u_i(s_{-i}^*(t_{-i}), a_i; t_i) P(t_{-i}|t_i)$$

### A First-Price Sealed-Bid Auction

Suppose there are two bidders, $i = 1, 2$.
The bidders' valuations $v_1$ and $v_2$ for a good are independently and uniformly distributed on $[0, 1]$.
Bidders submit their bids $b_1$ and $b_2$ simultaneously.
The higher bidder wins the good and pays his/her bidding price; the other bidder gets and pays nothing.
In the case that $b_1 = b_2$, the winner is determined by a flip of a coin.

1. What is the normal form of this game?

[1] Note that the players' strategy is a function from his type space to his action space.

[2] After the nature reveals the type to player $i$, he/she computes the new belief by using Bayes' rule $P(t_{-i}|t_i) = \frac{P(t_{-i}, t_i)}{P(t_i)}$.

2. There are many Bayesian Nash equilibria of the game (and many equilibria may not make realistic sense). We only look for equilibria in the form of *linear strategies* of the form:

$$b_1(v_1) = a_1 + c_1 v_1,\ b_2(v_2) = a_2 + c_2 v_2,$$

   where $0 \leq a_i < 1$, $c_i > 0$ for $i = 1, 2$. [3]

   Rationale of assumptions:

   - $a_i \geq 0$ reflects the fact that bids cannot be negative;
   - $c_i > 0$ implies high bids for high valuations.
   - If $a_i \geq 1$, then together with $c_i > 0$, it follows that $b_i(v_i) > v_i$ for all $v_i \in [0, 1]$, which is absurd.

   What is the Bayesian Nash equilibrium with linear strategies?

3. If you knew these two strategies are Bayesian Nash equilibrium, and only wish to prove it. Can you do it easily?

4. Can you guess a Bayesian Nash equilibrium in the case of $n$ bidders, and prove it?

5. What is the expected Auctioneer's revenue?

### A Second-Price Sealed-Bid Auction / Vickrey Auction)

There are $n$ potential bidders, with valuations $v_1, ..., v_n$ for a good.
The bidders valuations are independently and uniformly distributed on $[0, 1]$.
Bidders submit their bids $b_i$ simultaneously.
The highest bidder wins the good and pays the second highest bidding price; all the other bidders get and pays nothing.
In the case of a tie, the winner is determined uniformly randomly among all highest bidders.

1. What is the normal form of this game?

2. Consider the strategy $s_i^*(v_i) = v_i$ for player $i$. Show that no matter what other bidders' strategies, this strategy always maximizes the payoff $i$ function for player $i$. [4]

3. Conclude that $s^* = (s_1^*, ..., s_n^*)$ is a Bayesian Nash equilibrium.

4. What is the expected Auctioneer's revenue?

[3] We confine the search for equilibria to linear functions. However, this is not a restriction on the strategy space, which is still the set of all functions $b_i : [0, 1] \longrightarrow [0, \infty)$. This means that an equilibrium must be better than *all functions*!

[4] We say $s_i^*$ *(weakly) dominates* all the other strategies if this happens.

## A variation of First-Price Sealed-Bid Auction

A mechanism is called *incentive-compatible (IC)* if every participant can achieve the best outcome to themselves just by acting according to their true preferences.

We see that in the First-Price Sealed-Bid Auction does not satisfies the IC condition. Can you modify the auction so that it satisfies IC?

More generally, we have

**Theorem 1 (The Revelation Principle)** *Any Bayesian Nash equilibrium of any Bayesian game can be represented by a payoff-equivalent* [5] *incentive compatible direct mechanism*[6]*.*

## Generalizations

- How to design a mechanism with more items for bidding? (Vickrey-Clarke-Groves Auction)
- How to design a mechanism that maximizes the seller revenue? (The Revelation Principle, Bayesian-optimal mechanism, Revenue equivalence theorem)

[5] Two Bayesian games $G, \bar{G}$ are said to be *payoff-equivalent* if any Bayesian Nash equilibrium of $G$ can be represented by a Bayesian Nash equilibrium of $\bar{G}$ and their payoffs are identical.

[6] A direct mechanism is a particular type of Bayesian game. In such a game, the arbitrator designs a value function $v = (v_1, ..., v_n)$. The players of types $t = (t_1, ..., t_n)$ make claims about their types $\tau = (\tau_1, ..., \tau_n)$, and receive payoffs $v(\tau)$.