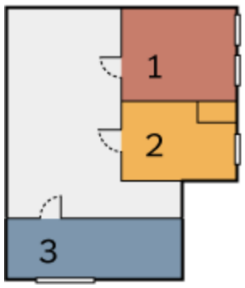
MATH 157: Mathematics in the world
Notes 23 (April 23, 2019)

Sperner's Lemma

To Divide the Rent, Start With a Triangle[1]

Three friends Ashwin, Bret and Chad want to share an apartment. The total rent is $3,000 but the rooms are different sizes. How can they choose rooms and divide the rent fairly?



Envy Free Cake Cutting (Approximation algorithm)

n people are sharing a (heterogeneous) cake. How can they divide the cake so that everyone feels that their allocated share is at least as good as any other share, according to their own subjective valuation.

Theorem 1 (Sperner's Lemma in dimension 2) *Given a triangle ABC, and a triangulation T of the triangle, the set S of vertices of T is colored with three colors in such a way that*

1. *A, B, and C are colored 1, 2, and 3 respectively.*
2. *Each vertex on an edge of ABC is to be colored only with one of the two colors of the ends of its edge. For example, each vertex on AC must have a color either 1 or 3.*

Then there exists a triangle from T, whose vertices are colored with the three different colors. More precisely, there must be an odd number of such triangles.

[1]This problem comes from a New York Times article, see https://www.nytimes.com/2014/04/29/science/to-divide-the-rent-start-with-a-triangle.html.

From Sperner's Lemma to Brouwer's fixed point theorem and Nash's Theorem

Theorem 2 (Brouwer's fixed point theorem in dimension 2) *Every continuous function from a closed disk to itself has at least one fixed point.*

Theorem 3 (Nash's existence theorem) *Every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium.*

References

[1] John Nash. *Non-Cooperative Games* The Annals of Mathematics, Second Series, Vol. 54, No. 2: 286-295, 1951.

[2] Francis E. Su. *Rental Harmony: Sperner's Lemma in Fair Division.* The American Mathematical Monthly. **106** (10): 930-942, 1999.