MATH 157: Mathematics in the world
Notes 24 (April 25, 2019)

The Problem

'Imagine yourself at a casino with $1,000. For some reason, you desperately need $10,000 by morning; anything less is worth nothing for your purpose. What ought you do? The only thing possible is to gamble away your last cent, if need be, in an attempt to reach the target sum of $10,000. There may be a moment of moral confusion and discouragement. For who has not been taught how wrong and futile it is to gamble, especially when short of funds? Yet, gamble you must or forgo all chance of the great purpose that can be achieved only at the price of $10,000 payable at dawn. The question is how to play, not whether.' [1]

Suppose a casino offers the following 2-player game of chance.

You begin with a balance of x_0 dollars, while the casino begins with $N - x_0$ dollars, where $0 < x_0 < N$.

At each round r of play you may wager any amount b_r, which the casino must match. Neither player may have a negative balance at any time, meaning $b_r \leq x_r$ and $b_r \leq N - x_r$. If you win the round, you take the pot $2b_r$, and if you lose, the casino takes the pot.

Your probability of winning each round is a fixed value $P \in (0, 0.5)$ which remains constant throughout the course of the game and is independent of the result of any other round.

The game ends as soon as either player's balance reaches zero ($x_r = 0$ or $N - x_r = 0$).

Define a wagering strategy as a list $s = (b_0, b_1, \ldots, b_N)$, such that if your balance is x dollars, you will wager b_x dollars. Notice that b_0 and b_N are clearly zero, because the game must be over at that point. For example, your strategy might be to bet $b_x = \$2$ whenever x is even and $b_x = \$1$ otherwise.

For any strategy s, let q_{x_0} be the probability that a player following strategy s and given a starting balance of x_0 will eventually beat the casino. Let the strategy's dollar weight be the arithmetic mean

$$W_s = \frac{1}{N+1} \sum_{x=0}^{N} b_x.$$

A strategy s may then be considered **optimal** if the overall probability $q_x \geq \tilde{q}_x$ for all x and all wining probability $\tilde{q}$ of any other strategies. If more than one strategy can be optimal, then s is **minimal** if

[1] Quoted from 'Inequalities for stochastic processes (how to gamble if you must)')

the dollar weight W_s is the least among all optimal strategies.

Assume the that all the dollar amounts are integer, and assume you know P prior to designing your strategy.

Given the above constraints, find the **minimal optimal** strategy, and compute the corresponding winning probability given a starting balance x, that will depend also on the total dollar amount N and the winning probability P.

For concreteness, we will assume $N = 32$, and $P = 0.2$.