MATH 157: Mathematics in the world

Notes 6 (February 14, 2019)

Problems

Covering with dominoes

1. We are given a 8×8 chess board with two diagonal corners missing. Is it possible to cover it completely with 2×1 dominoes without any pieces overlapping?

2. We are given a 9×9 board with a corner missing. Is it possible to cover it completely with dominoes?

3. Suppose now we are given a 9×9 board with one square adjacent to the corner (sharing one side) missing. Is it possible to cover it completely with dominoes?

Packing bricks

Is it possible to fit 53 bricks of size $1 \times 1 \times 4$ into a $6 \times 6 \times 6$ box?

Packing bricks - 2

Is it possible to cover a 101×101 board with only 2×2 and 3×3 squares?

Domino covering again

Invent a connected shape made out of squares on the square grid that cannot be cut into dominoes, but if you add a domino to the shape then you can cut the new bigger shape.

Handy sums

If you have never seem the following expressions, please look them up. My favorite proofs use basic geometry, but there are many other ones.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} 2i - 1 = n^2, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Sums

Evaluate the following sums:

$$\sum_{i=1}^{n} i^2 + i, \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} i \cdot j^2, \qquad \sum_{i=1}^{n}\sum_{j=1}^{3} i^j, \qquad \sum_{i=1}^{n^2} \left\lfloor \sqrt{i} \right\rfloor.$$

Divisibility criteria

A *divisibility criterion* for a number n is a rule which quickly allows us to check if a number is divisible by n. Of course, one can always divide into n and verify that that there is no remainder, but that is a very tedious operation.

We are looking for something much quicker, a condition one can almost check in a single glance. Usually these tricks are expressed in terms of the digits in decimal expansion. For example, a number is divisible by 10 if it ends in a 0 (in its decimal expansion).

1. Find a divisibility criterion for 2.
2. Find a divisibility criterion for 5.
3. Find a divisibility criterion for 4.
4. Find a divisibility criterion for 8.
5. Find a divisibility criterion for 3.
6. Find a divisibility criterion for 9.
7. Find a divisibility criterion for 11.

Divisibility criteria - 2

1. Prove that the following is a divisibility criterion for 7. Split off the last digit. Double it and subtract that from the number that is left. If the result is divisible by 7 then so is the original number.
   Note: you may need to repeat the process several times to get a number small enough to be able to say if it is divisible by 7.
2. Find and proof a divisibility criterion for 13.

All ones

Find the smallest positive integer n such that the decimal expansion of $99n$ contains only ones.

Missing digits

1. The number $62ab427$ is a multiple of 99. Find the digits a and b.
2. Suppose $33! = 8ab331761881188649551819440128cd00000$. Find the digits a, b, c, and d.

Trailing zeroes

1. How many zeroes does 100! end in?
2. Write a formula for the number of trailing zeroes in the number n!

Sum the digits!

Let $S(n)$ denote the sum of the decimal digits of n. Consider the following recursive sequence

$$a_0 = 2015!, \qquad a_{n+1} = S(a_n).$$

What is the behavior of the sequence for large n? Can you find its limit? How soon is it achieved? For an extra credit, estimate the number of steps it takes to reach the limit.

Counting divisors

What is the smallest number which has exactly 77 divisors?

The locker game

The students of a local high school enjoy the following extra-curricular activity. After the end of the day, all 100 students go in front of their closed lockers, numbered 1 to 100. The headmaster then proceeds to blow a whistle 100 times. At the first sound, all students open their lockers. At the second, the students standing by even lockers close them. The game continues in a similar manner – at the n-th step students in front of lockers divisible by n toggle them.

Which lockers are open at the end of the game?