MATH 157: Mathematics in the world
Notes 7 (February 19, 2019)

Problems

Divisibility criteria

1. Find a divisibility criterion for 9.
2. Find a divisibility criterion for 11.

Divisibility criteria - 2

1. Prove that the following is a divisibility criterion for 7. Split off the last digit. Double it and subtract that from the number that is left. If the result is divisible by 7 then so is the original number.
   Note: you may need to repeat the process several times to get a number small enough to be able to say if it is divisible by 7.
2. How about divisibility criterion for 13? (You will be guided to find this in your homework!)

Trailing zeroes

1. How many zeroes does 100! end in?
2. Write a formula for the number of trailing zeroes in the number $n!$

Sum the digits!

Let $S(n)$ denote the sum of the decimal digits of n. Consider the following recursive sequence

$$a_0 = 2015!, \qquad a_{n+1} = S(a_n).$$

What is the behavior of the sequence for large n? Can you find its limit? How soon is it achieved?

For an extra credit, estimate the number of steps it takes to reach the limit. (You will be guided to find this number in your homework!)

The last digit

What is the last digit of 7^{2015} (in base 10)?

Not a prime

Show that $3^{594} + 1$ is not a prime.

Prime numbers and factorization

Fact: every positive integer can be written in a unique way as a product of prime numbers.

Euclid's Theorem

How do we know that there are infinitely many primes?

Primality checking

How can you check that a number n is prime? What is the fastest algorithm you can find?

Sieve of Eratosthenes

Given a number n, how can you find all prime numbers $p \leq n$?

Euclid's algorithms

Given two numbers a and b, how can you compute their greatest common divisor (a, b) fast?

Extra

All ones

Find the smallest positive integer n such that the decimal expansion of $99n$ contains only ones.

Counting divisors

What is the smallest number which has exactly 77 divisors?

Missing digits

1. The number $62ab427$ is a multiple of 99. Find the digits a and b.
2. Suppose $33! = 8ab33176188118864955181944012 8cd00000$. Find the digits a, b, c, and d.

The locker game

The students of a local high school enjoy the following extra-curricular activity. After the end of the day, all 100 students go in front of their closed lockers, numbered 1 to 100. The headmaster then proceeds to blow a whistle 100 times. At the first sound, all students open their lockers. At the second, the students standing by even lockers close them. The game continues in a similar manner – at the n-th step students in front of lockers divisible by n toggle them.

Which lockers are open at the end of the game?