

MATH 157: Mathematics in the world

Notes 8 (February 21, 2019)

Prime numbers and factorization

Fact: every positive integer can be written in a unique way as a product of prime numbers.

Euclid's Theorem

How do we know that there are infinitely many primes?

Primality checking

How can you check that a number n is prime? What is the fastest algorithm you can find?

GCD and LCM

GCD is an abbreviation for greatest common divisor, and LCM for least common multiple. When no confusion can arise, we will denote these functions by

$$(a, b) = \gcd(a, b), \quad [a, b] = \text{lcm}(a, b).$$

The following properties are worth noting:

- $(a, b) = (b, a)$, $[a, b] = [b, a]$,
- $(a, b) = (-a, b)$,
- $(a, b) = (a, b \pm a)$.

If $(a, b) = 1$, we call a and b *coprime*.

Euclid's algorithms

Given two numbers a and b , how can you compute their greatest common divisor (a, b) fast?

Fibonacci numbers and GCD

Find the greatest common divisor of F_{2019} and F_{2020} .

Finding coprimes

Let n be a positive integer. If $S \subset \{1, \dots, 2n\}$ is a set of size $n + 1$, show that there exist two distinct coprime elements $a, b \in S$.

GCD and LCM theorem

Prove that we have $a, b = ab$. Can you find similar identities for 3 or more numbers a, b, c ?

The last digit of Fibonacci numbers

Consider the sequence of last digits (in base 10) obtained from the Fibonacci sequence. Is it periodic and why? How long is its period?

Congruences and powers

Smallest sum of digits

What's the smallest sum of digits that a multiple of 29 can have?

Two more than a square of a prime

What are the possible remainders that a square can have when it is divided by 3? Can a prime be of the form $p^2 + 2$, where p is also prime? How many can you find?

Difference of squares

Is it possible for the difference of two squares to be 6? And for it to be 12?

One less than a perfect square

Find a prime number which is one less than a perfect square. How many can you find?

One more than a perfect square

Find a prime number which is one more than a perfect square. How many can you find?

Fermat's Little Theorem

Powers and primes

What are the possible remainders that a fourth power can have mod 5? How do you think this could generalize to any prime number p ?

Extra

Digits strikes

A multidigit number contains a single digit zero. When you strike it, the number becomes 9 times smaller. At which position was the zero located? What are all possible such numbers?

A diophantine equations*

Find all integer solutions of $x^3 + 5y^3 + 25z^3 = 5xyz$.