options markets

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preface

The trading of options and the scientific study of options both have long histories, yet both underwent revolutionary changes at virtually the same time in the early 1970s. These changes, and the subsequent events to which they led, have greatly increased the practical value of a thorough understanding of options.

Although options have been traded for centuries, they were, until recently, relatively obscure and unimportant financial instruments. Options markets were fragmented, and transactions were both costly and difficult to arrange. All of this changed in 1973 with the creation of the Chicago Board Options Exchange, the first registered securities exchange for the purpose of trading in options. The exchange began modestly, with trading only in call options on sixteen common stocks, but it soon became a tremendous success. This success, in turn, led to a series of innovations in options trading. In 1975 and 1976, the American, Philadelphia, and Pacific Stock Exchanges began trading call options on common stocks. In 1977, put options were also listed on all of the exchanges. Within ten years of the establishment of the first options exchanges, the volume of trading in stock options grew to a level often exceeding, in terms of share equivalents, that of the New York Stock Exchange, and the number of stocks on which exchange-listed options were available rose from sixteen to nearly four hundred. In the early 1980s, listed options trading has also expanded into financial instruments other than common stocks, including options on stock indexes, foreign currencies, U.S. Treasury securities, stock index futures, commodity futures, U.S. Treasury bond futures, and foreign currency futures. It certainly seems reasonable to predict that options will play an increasingly important role in financial markets in the future.

The study of options has an illustrious history dating back to the late nineteenth century, and some of the early works in the field made pioneering contributions to the theory of stochastic processes. Nevertheless, a completely satisfactory theory of option valuation was not developed until the early 1970s. Since that time, option pricing theory has been refined and expanded in many ways and has proven to be extremely useful. Indeed, its implications have extended well beyond exchange-traded options. Quite generally, an option contract can be thought of as any security whose returns are contractually related to the returns on some other security or group of securities. From this perspective, the principles of option valuation can be applied to a broad range of financial instruments. Included in this range are not only securities that have obvious option-like features, such as warrants and convertible bonds, but also securities that do not seem to be like options at all, such as common stocks and ordinary bonds.

The purpose of our book is to provide a detailed discussion of these academic and institutional developments. Rather than give superficial coverage of many different kinds of options, we have chosen to concentrate on the oldest and largest segment of the options markets, options on common stocks. As will be evident, most of our conclusions reached in the context of stock options can be easily modified to apply to other types of options.

Chapter 1 gives a basic introduction to put and call contracts and the market structure in which they are traded. This chapter also shows the profit-and-loss consequences of some elementary option trading strategies. In Chapter 2, we examine the fundamental variables that affect option value and discuss why investors use options. Chapter 3 continues the introduction to market structure given in Chapter 1 and provides a detailed description of the environment in which stock options are traded.

Chapter 4 begins our material on option valuation. Here we derive certain basic properties that option prices must satisfy if there are to be no arbitrage opportunities. These methods require virtually no knowledge of stock price movements, and are consequently quite general. In Chapter 5, we describe how additional information about stock price movements can be used to derive the exact value of an option. In Chapter 6, we show how to modify and implement the theory developed in Chapter 5 for practical investment purposes. Chapter 7 provides some further extensions of option valuation and shows how the theory can be applied to corporate securities. Chapter 8 discusses the social role of options and gives suggestions for improving options markets, some of which have already been implemented.

We owe a substantial debt to our academic colleagues who have contributed to the strong theoretical basis for option valuation now available. For each subject, we have usually given only one or two primary references, but many additional references can be found in the bibliography. Special thanks go to Fischer Black and Myron Scholes, who developed the first completely satisfactory theory of option valuation; to Robert Merton, who extended the theory in fundamental ways and provided many other insights into option pricing; and to William Sharpe, who discovered a way to derive the basic principles of option valuation using only elementary mathematics. We have also benefited greatly from the comments and suggestions of a number of other individuals, including Michael Brennan, George Constantinides, Kenneth Dunn, John Ezell, Gary Gastineau, Robert Geske, Steven Givot, Blair Hull, Jonathan Ingersoll, Hayne Leland, Louis Morgan, Krishna

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John C. Cox
Mark Rubinstein

introduction

1

1-1. WHAT ARE PUTS AND CALLS?

Every field has its own special vocabulary. Since options trading is no exception, we will begin with some basic definitions. Options markets exist or are planned for a wide variety of instruments, so to avoid needless repetition we will focus on the oldest and largest of these markets, options on common stocks.

> **A *call* option is a contract giving its owner the right to *buy* a fixed number of shares of a specified common stock at a fixed price at any time on or before a given date.**

The act of making this transaction is referred to as *exercising* the option. The specified stock is known as the *underlying security*. The fixed price is termed the exercise price or *striking price*, and the given date, the maturity date or *expiration date*. The individual who creates or issues a call is termed the seller or *writer*,[1] and the individual who purchases a call is termed the holder or *buyer*. The market price of the call is termed the premium or *call*

[1] The term *writer* is preferred to *seller* to emphasize the fact that, unlike stock which originates with a corporation, a call is literally issued on behalf of a single individual investor.

price. In other words, if a call is exercised, the complete transaction involves an exchange of

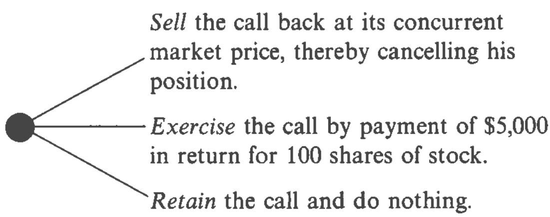
call price from buyer
for
call from writer

and a subsequent exchange of

striking price + call from buyer
for
common stock from writer.

The buyer has the right, but not the obligation, to make this subsequent exchange, so it will take place only if he feels it is in his best interest.[2]

For example, an ALCOA/JAN/50 call bought on the Chicago Board Options Exchange at the close of trading on August 20, 1979, would have cost $750, exclusive of commissions. This call gave the buyer the right to purchase 100 common shares of Alcoa stock for $50 per share at any time until January 18, 1980. On any trading day until the expiration date, the buyer can do one of three things:



On the expiration date itself, the third alternative is equivalent to permitting the call to *expire*. On this date, ignoring the sell alternative, it is easy to see which of the other two alternatives is in the best interest of the buyer. This depends on the concurrent price of Alcoa's common stock. If the stock price is greater than $50 per share, then (neglecting commissions), it will pay the call buyer to exercise the call, since by doing so he can buy the stock for $50 and, if he desires, immediately resell it on the market at a profit. On the other hand, if the stock price is less than $50, then the call buyer should let the call expire. Of course, after the expiration date, the call will be worthless, since the first two alternatives will have lapsed.

[2] A call should not be confused with a forward contract. At its maturity, a forward contract *must* be exercised. For a more detailed comparison of options with forward contracts, see the appendix to Chapter 2.

To represent the contractual implications of a call on its expiration date in symbols, let

K = the striking price,
S^* = the market price of the underlying security on the expiration date, and
C^* = the value of the call (to one share) on its expiration date.

Then

$$C^* = \begin{cases} S^* - K & \text{if} \quad S^* > K \\ 0 & \text{if} \quad S^* \leq K \end{cases}$$

or, alternatively, $C^* = \max[0, S^* - K]$. Some race-track terms have slipped into the options vocabulary: if $S^* > K$, the call is said to "finish in-the-money"; if $S^* < K$, the call is said to "finish out-of-the-money."

In contrast to a call,

> **A *put* option is a contract giving its owner the right to *sell* a fixed number of shares of a specified common stock at a fixed price at any time on or before a given date.**

If a put is exercised, the complete transaction involves an exchange of

put price from buyer
for
put from writer

and a subsequent exchange of

common stock + put from buyer
for
striking price from writer.

Again, this subsequent exchange will take place only at the choice of the buyer. If P^* is the value of a put (to one share) on its expiration date, then

$$P^* = \begin{cases} 0 & \text{if} \quad S^* \geq K \\ K - S^* & \text{if} \quad S^* < K \end{cases}$$

or, alternatively, $P^* = \max[0, K - S^*]$. In this case, if $S^* < K$, the put finishes in-the-money.

There are thus two *types* of options—puts and calls. All option contracts of the same type written on the same underlying stock constitute a *class* of options. Call and put options on the same underlying security are considered separate classes. Within a given class, all option contracts with the same expiration date and striking price constitute an option *series.*

Puts and calls are the basic forms of options. However, many other securities, such as corporate bonds and stocks, have similar characteristics. Thus much of what we have to say about puts and calls applies also to other types of securities. In Chapter 7 we will take advantage of this correspondence and develop relative pricing relationships among differing claims against the assets of a corporation.

It is easy to determine the value of a put or call on its expiration date. Finding its value at any time prior to expiration is much more difficult. We will provide an informal analysis in Chapter 2 and some precise answers in Chapters 4, 5 and 7. Taking a first cut now, it should be obvious that a put or call, since it can be exercised at any time until its expiration date, must be worth at least its current exercise value. Let

S = the current market price of the underlying security,
C = the current value of an associated call, and
P = the current value of an associated put.

The current value C of a call must then at least equal $\max[0, S - K]$, and the current value P of a put must at least equal $\max[0, K - S]$. We will refer to $\max[0, S - K]$ and $\max[0, K - S]$ as the exercise value or *parity value* of a call and a put, respectively. Since an option may be more valuable retained than exercised, its price may exceed its parity value.[3] We term this difference, $C - \max[0, S - K]$ for a call and $P - \max[0, K - S]$ for a put, the *premium over parity* of an option. The market price of an option is then equal to the sum of its parity value and premium over parity.[4]

The race-track terms mentioned earlier are applied before expiration as well: If $S > K$, a call is *in-the-money*; if $S = K$, it is *at-the-money*; if $S < K$, it is *out-of-the-money*. For a put, the definitions are reversed; a put is in-the-money if $S < K$. Thus, an option is in-the-money if its parity value is positive. If S is much greater than K, then a call is said to be *deep-in-the-*

[3] We will refer to exercise before the expiration date as *early exercise*; this does not imply that such exercising is inappropriate or untimely.

[4] The use of the term "premium" to refer to the total price of an option is a carryover from former times when options were almost always sold at-the-money (that is, $K = S$). Consequently, at the time of sale the "premium" and "premium over parity" were the same thing. To prevent possible confusion, we will always use the term "price" in place of premium, and reserve the term "premium over parity" to refer to the excess of the option price over the difference between the stock price and the striking price.

money and a put *deep-out-of-the-money.* If S is much less than K, then a call is *deep-out-of-the-money* and a put is *deep-in-the-money.*

The profit and loss implications of an option position are often confusing at first. *Payoff diagrams*, relating the profit from a position *if held to expiration* to the underlying stock price *at expiration*, are a useful aid. The most elementary payoff diagram (Fig. 1-1) describes a long position in stock. This relates the net profit realized on a given date in the future to the stock price on that date.

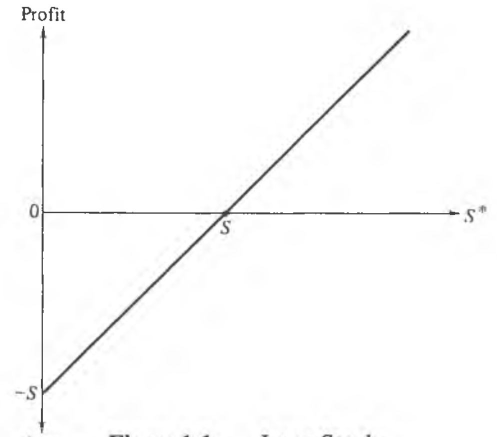


Figure 1-1 Long Stock

If the stock price on the final date is zero (that is, $S^* = 0$), then a long stock position will have experienced a net loss of S, where S is the current stock price. If $S^* = S$, the position will result in no profit or loss. In general, net profit will equal $S^* - S$. A \$1 increase in S^* is exactly matched by a \$1 increase in net profit. In brief, the *payoff line* for this position is a 45° line with positive slope, with a zero profit point at S. For simplicity, we have so far ignored complications that may be created by commissions, margin, taxes, and dividends. Similarly, a short position[5] in the stock is described by Figure 1-2. With a long position in the stock, the possible loss is limited to S, while the possible gain is unlimited. With a short position, just the reverse is true. Here the possible gain is limited to S, while the possible loss is unlimited.

---

[5] A short position involves selling stock one does not own. This is accomplished by borrowing the stock from an investor who has purchased it. At some subsequent date, the short seller is obligated to buy the stock to pay back the lender of the shares. Since this repayment requires *equal shares rather than equal dollars*, the short seller benefits from a decline in the stock price.

If cash dividends are paid on the stock while a short position is maintained, these are paid to the buying party of the short sale. The short seller must also compensate the investor from whom the stock was borrowed by matching the cash dividends from his own resources. In brief, not only does the short seller not receive cash dividends, but he must also make matching payments.

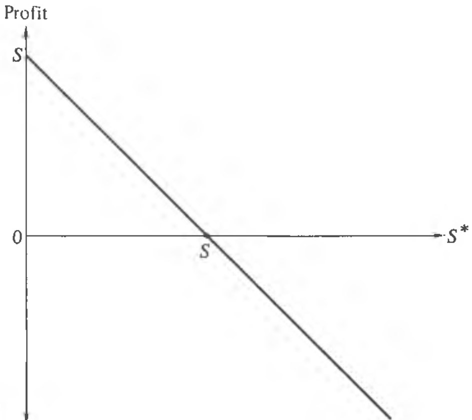


Figure 1-2 Short Stock

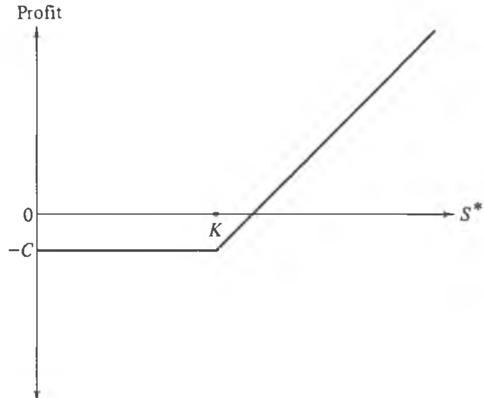


Figure 1-3 Buy Call

Figures 1-3, 1-4, 1-5, and 1-6 show the profit-and-loss implications of the four basic positions in options. Evidently, a purchased call is like a long position in the underlying security, except that it has the advantage of insurance against extreme downside movements in the stock price. Similarly, a purchased put is like a short position in the underlying security, except that it affords protection against extreme upside movements in the stock. However, both puts and calls have an important disadvantage: The insurance they provide costs money in the form of the premium over parity. To emphasize this point, suppose the current stock price S is equal to the striking price K and by the expiration date the stock price remains unchanged, so that $S^* = S$. While a long or short position in the stock would show a zero net profit, a purchase of a put or a call would result in the loss of the entire investment, the current option price.

At first glance, it seems that a stop-loss order would provide the same kind of insurance as a call. Suppose that we purchase a share of the stock

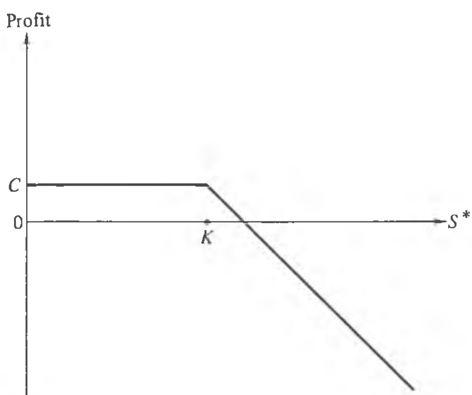


Figure 1-4 Write Call

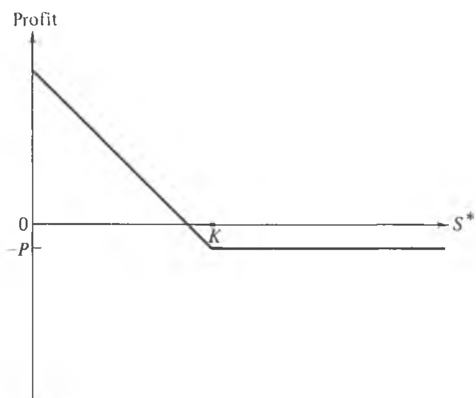


Figure 1-5 Buy Put

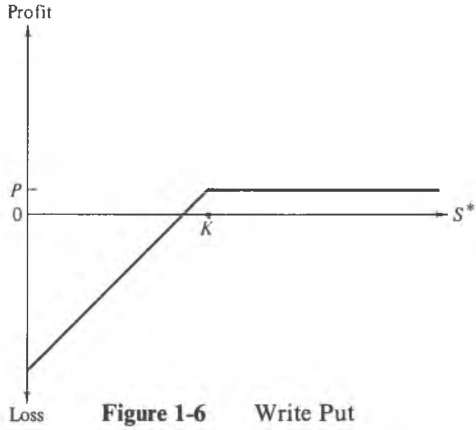


Figure 1-6 Write Put

by investing $S - K$ and borrowing the remainder, K. At the same time we tell our broker to sell the stock if its price drops to K and place the proceeds in default-free bonds. To make things simpler, suppose interest rates and dividends are zero. The value of our position on any specified expiration date is, after repayment of the borrowing, apparently exactly the same as a call—0 if $S^* \leq K$ and $S^* - K$ if $S^* > K$. Furthermore, the net investment required was only $S - K$. But this argument has a fatal flaw. A call will indeed receive $S^* - K$ whenever $S^* > K$, but we will receive this amount only if $S^* > K$ *and* the stock price never goes below K before the expiration date. Furthermore, it may not be possible to execute the stop-loss order exactly at K. If the stock price can take sudden jumps, we may end up being sold out at a price below K. In that event, we would not receive enough from the sale to cover our borrowing. Not only would we lose our initial investment, but also we would have to put up more money to close out the position. This can never happen with a call. Consequently, the owner of a call may receive some gains that we will miss, and he will avoid some losses that we may have to bear. These advantages make a call worth more than its parity value. As we will see in the next chapter, when dividends and interest rates are not zero, they will also influence the premium over parity. Hence, we can conclude that an option gives a kind of insurance that cannot be obtained with a stop-loss order (or, in the case of a put, with a contingent buy order).

The payoff diagrams clearly illustrate an important point: *the options market is a zero-sum game.* That is, what the option buyer profits, the option writer loses, and vice versa. Ignoring the impact of taxes, any claim that the options market is typically profitable for both buyers and writers cannot be correct. Indeed, considering commissions, it could, on average, be simultaneously unprofitable for both groups and must be unprofitable for both groups taken together. Nevertheless, the welfare of both groups can be improved by trading in options. Other forms of insurance, for example, are also zero-sum games, but no one would argue that they do not have an important economic function.

The great flexibility afforded by puts and calls only becomes evident when combined positions are considered, such as writing a call against a long position in the stock, or simultaneously writing and buying different calls on the same underlying stock. In the next section, payoff diagrams are used to analyze these more complex "covered" positions. In Chapter 6, these diagrams are further generalized to analyze the potential profit from a position prior to expiration.

1-2. PAYOFF DIAGRAMS FOR ELEMENTARY STRATEGIES

If the only securities to be bought or sold are puts and calls on the same underlying security and the underlying security itself, then there are four elementary types of positions that can be taken:

1. Uncovered
2. Hedge
3. Spread
4. Combination

The six *uncovered* or "naked" *positions*—long stock, short stock, buy call, write call, buy put, write put—have already been examined. These were shown to give rise to relatively simple payoff lines. Hedges, spreads, and combinations are types of *covered positions*, in which one or more securities protect the returns of one or more other securities, all related to the same underlying stock.

> **A *hedge* combines an option with its underlying stock in such a way that either the stock protects the option against loss or the option protects the stock against loss.**

In other words, a hedge combines a long position in the stock with a written position in calls or a purchased position in puts; a "reverse hedge" combines a short position in the stock with a purchased position in calls or a written position in puts. The most popular hedge consists of writing one call against each share owned of the underlying stock. To analyze this, as for all covered positions, we superimpose the relevant separate payoff diagrams—long stock and write call. The payoff line for the combined position is determined, for each value S^* of the stock at expiration, by adding together the vertical distances of the two separate payoff lines from the horizontal axes. A comparison of Figures 1-6 and 1-7 shows that the payoff diagram for writing a call and buying the stock has the same shape as the payoff diagram for writing a put. We might now suspect that this result could be used to find a relationship between put and call values, and we will soon see that this is indeed true.

Slight variations of the one-to-one hedge produce new payoff patterns. A "ratio" hedge might involve two calls written against each share of stock. As shown in Figure 1-8, this combined position creates a "payoff triangle" that produces a profit as long as the stock price does not experience an extreme change in either direction. However, suppose you, as a potential investor, believe some important news is about to be made public (such as the results of a merger negotiation) that would have a significant impact on the market price of a stock. But you do not know in advance whether the news will be favorable or unfavorable. The "reverse hedge" (Figure 1-9)—buy two calls against each shorted share of the stock—might be an appropriate position. In this case, you show a profit only if the stock price makes a strong move—and it does not matter in which direction! Without options it may be difficult for you to take proper advantage of your beliefs. This is a clear example, among many others, where the availability of options adds flexibility to investment decisions.

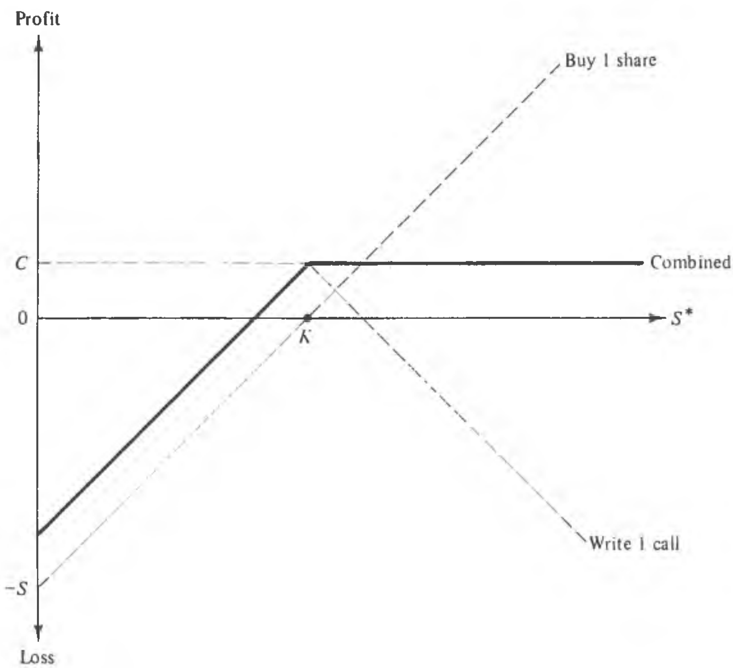


Figure 1-7 1 : 1 Hedge ($S = K$)

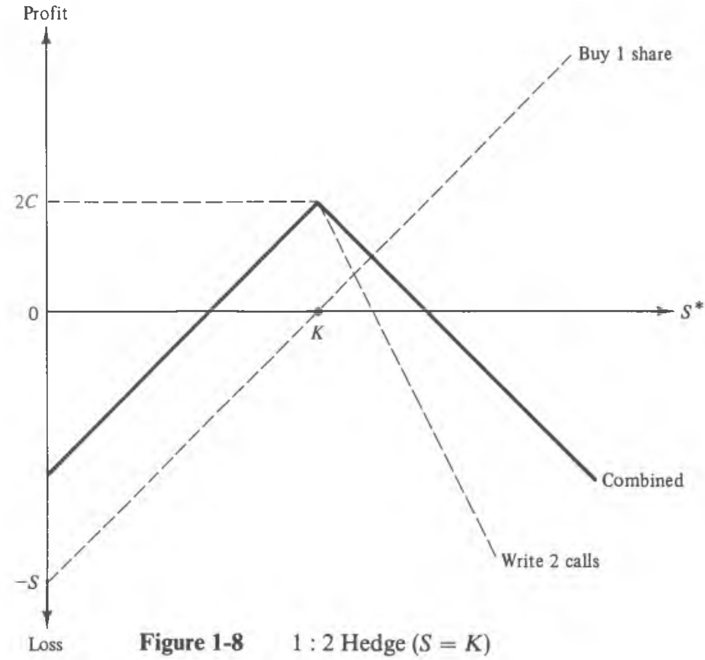


Figure 1-8 1 : 2 Hedge ($S = K$)

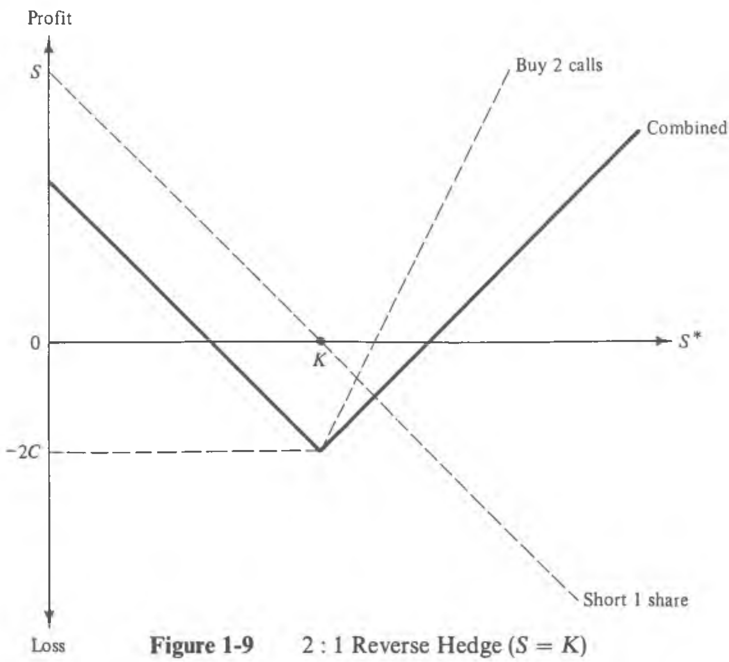


Figure 1-9 2 : 1 Reverse Hedge ($S = K$)

Figures 1-10, 1-11, and 1-12 show some similar hedges using puts.

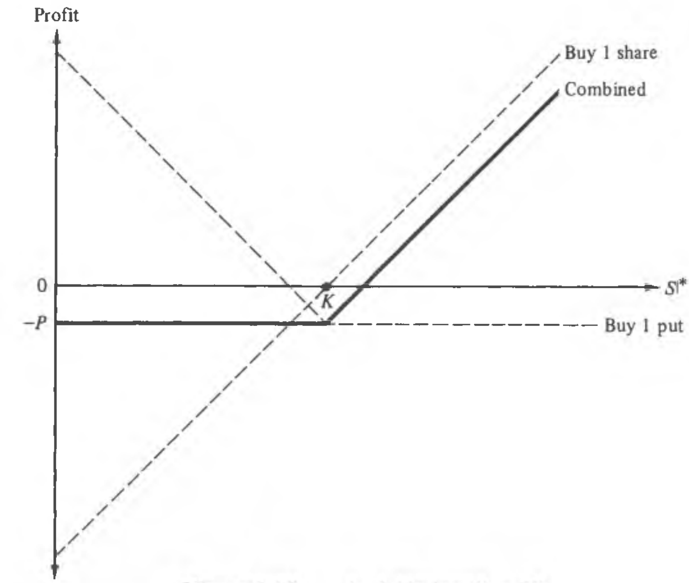


Figure 1-10 1 : 1 Hedge ($S = K$)

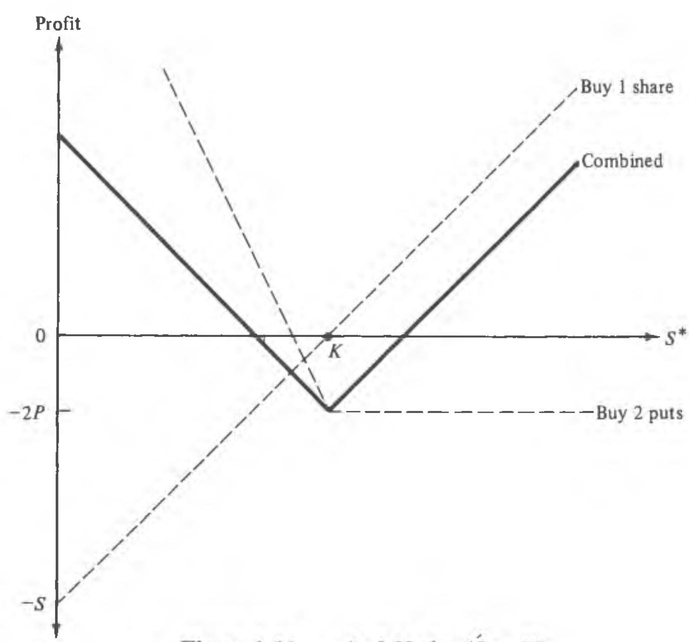


Figure 1-11 1 : 2 Hedge ($S = K$)

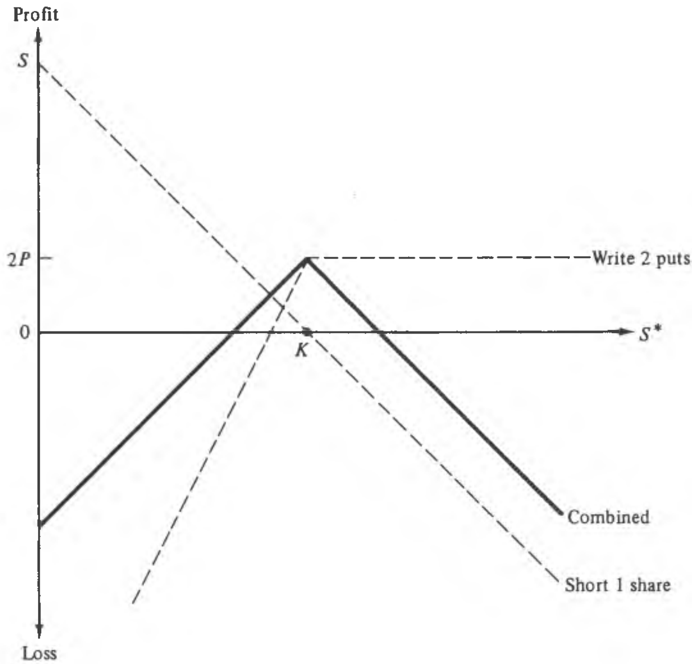


Figure 1-12 2 : 1 Reverse Hedge ($S = K$)

A *spread* combines options of different series but of the same class, where some are bought and others are written.

Two common spreads are the *vertical spread* (also termed "money," "price," or "perpendicular" spread) and the *horizontal spread* (also termed "time" or "calendar" spread). As shown in Table 1-1, in a vertical spread, one option is bought and another sold, both on the same underlying stock and with the same expiration date, but with *different striking prices.*

Table 1-1
VERTICAL SPREAD (CALLS)

Striking Price	*Expiration Month*			*Stock Price*
	JAN	*APR*	*JUL*	
XYZ 30	11	$12\frac{3}{8}$	$13\frac{1}{2}$	40
XYZ 40	$4\frac{1}{2}$	$6\frac{1}{4}$	$7\frac{7}{8}$	40
XYZ 50	$1\frac{1}{4}$	$2\frac{7}{8}$	$4\frac{3}{8}$	40

Table 1-2
HORIZONTAL SPREAD (CALLS)

Striking Price	Expiration Month JAN	APR	JUL	Stock Price
XYZ 30	11	$12\frac{3}{8}$	$13\frac{1}{2}$	40
XYZ 40	$4\frac{1}{2}$	$6\frac{1}{4}$	$7\frac{7}{8}$	40
XYZ 50	$1\frac{1}{4}$	$2\frac{7}{8}$	$4\frac{3}{8}$	40

As shown in Table 1-2, in a horizontal spread, one option is bought and another sold, both on the same underlying stock and with the same striking price, but with *different expiration dates.*

Note that horizontal spreads cannot be represented by a standard payoff diagram. Later, when we develop exact formulas for pricing options, it will be possible to represent horizontal spreads by payoff diagrams, since we will know the value of the long-maturity option on the expiration date of the short-maturity option for each level of the stock price on that expiration date.

In a *diagonal spread,* one option is bought and another sold, both on the same underlying stock, but with *different striking prices* and *different expiration dates.* By extension, there are four types of diagonal spreads. Again, these cannot be graphed in the usual payoff diagram because they involve options of different maturity. The terms vertical, horizontal, and diagonal arise from the format for listing put and call prices in the newspaper.

Each of the spreads has its *bullish* and *bearish* versions. In a bullish vertical spread, the option purchased has the lower striking price. For a bullish horizontal spread, the option with the longer time to expiration is purchased. Finally, with a bullish diagonal spread, the purchased option has both a lower striking price and a longer time to expiration than the written option. For the corresponding bearish spreads, the positions are reversed. Although these terms are frequently used, they can be quite misleading. The names imply that a bullish spread should benefit from an increase in the stock price and that a bearish spread should benefit from a decrease. Unfortunately, as we will see in Chapter 6, this is not always true for horizontal and diagonal spreads. The implication is correct only for vertical spreads.

Figure 1-13 is the payoff diagram for a bullish vertical spread. The call with the lower striking price has been purchased and the call with the higher striking price has been written. In Table 1-1 we would have bought the XYZ/30 for $13\frac{1}{2}$ and sold the XYZ/50 for $4\frac{3}{8}$. Our spread requires an

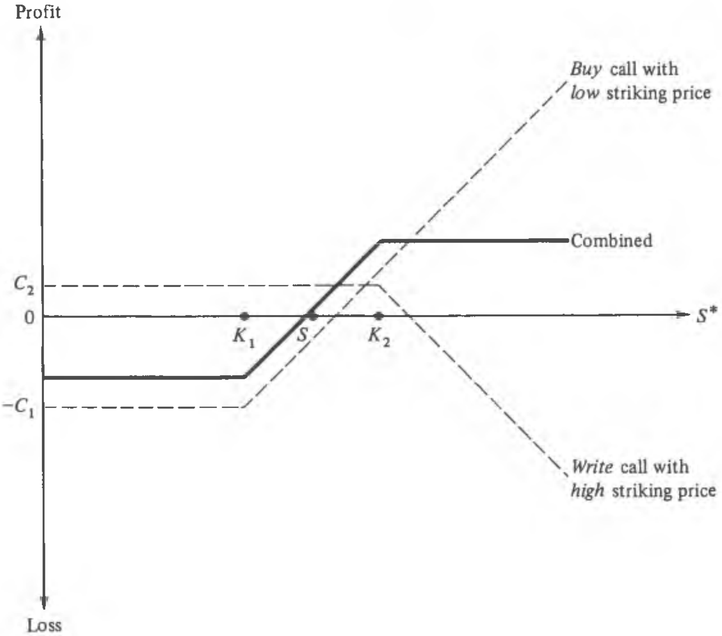


Figure 1-13 Bullish Vertical Spread

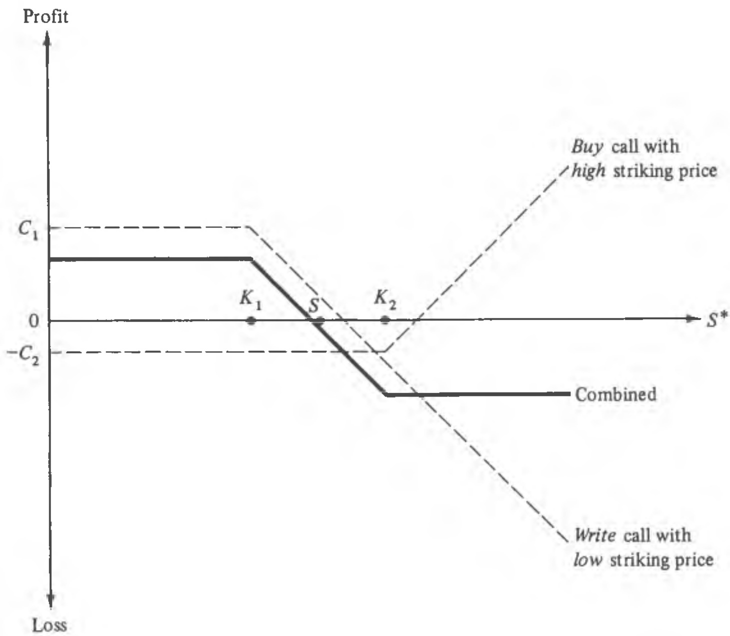


Figure 1-14 Bearish Vertical Spread

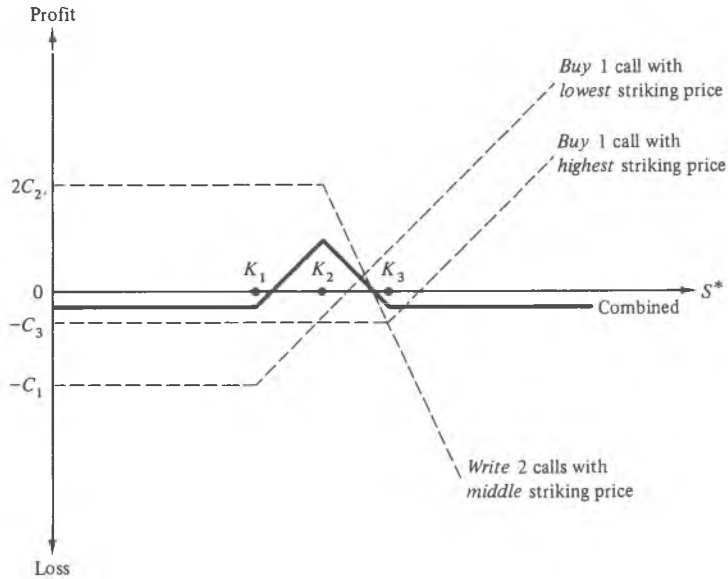


Figure 1-15 Butterfly Spread ($S = K_2$)

initial cash outlay. This suggests why a bullish spread is alternatively termed a *purchased spread*. We would be said to have "bought the spread."

On the other hand, had the XYZ/30 been sold and the XYZ/50 bought, we would have a bearish or *written spread*. These designations are actually more appropriate than the terms bullish and bearish. As we will see in Chapter 4, neglecting margin, all of the spreads described as bullish should, for calls, indeed require initial cash outlays. Similarly, the bearish positions should produce an initial cash inflow. The terms *purchased* and *written* are descriptively accurate, while the terms *bullish* and *bearish* are not. One disadvantage is that the labels must be reversed for puts. A bullish spread using puts will be a written spread; the corresponding bearish spread will be a purchased spread.

In a *butterfly spread*, two options in the middle, with respect to striking price or expiration date, are purchased (written) against writing (buying) one option on each side, all on the same underlying stock. Figure 1-15 illustrates a butterfly vertical spread, where the middle calls have been written and the end calls purchased. A small profit would be realized only if the stock price stays near the striking price of the written calls. A butterfly spread does not qualify as an "elementary" position because it can be interpreted as a portfolio of a bearish and bullish vertical spread, or a portfolio of a bearish and bullish horizontal spread.

Figures 1-16, 1-17, and 1-18 show some similar spreads using puts.

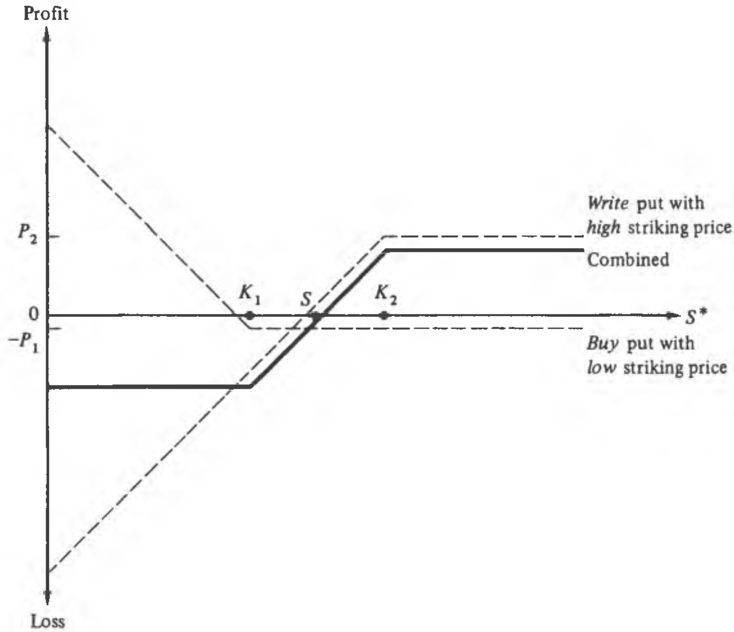


Figure 1-16 Bullish Vertical Spread

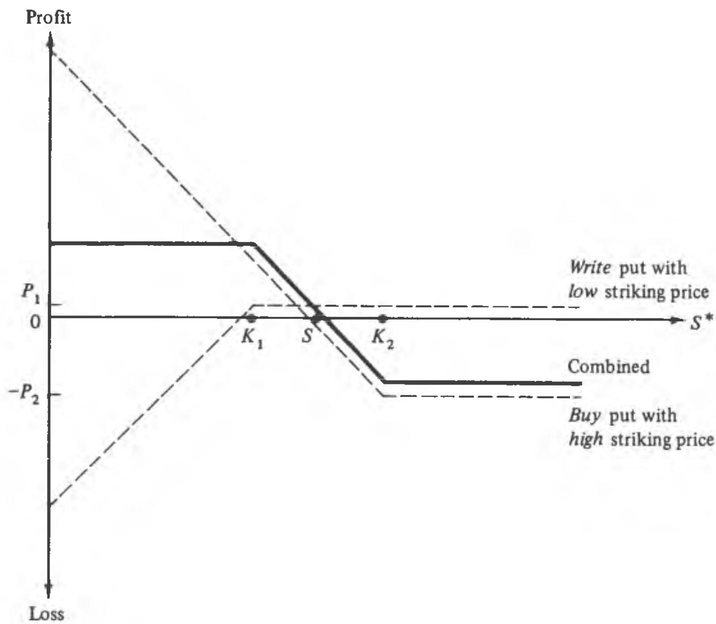


Figure 1-17 Bearish Vertical Spread

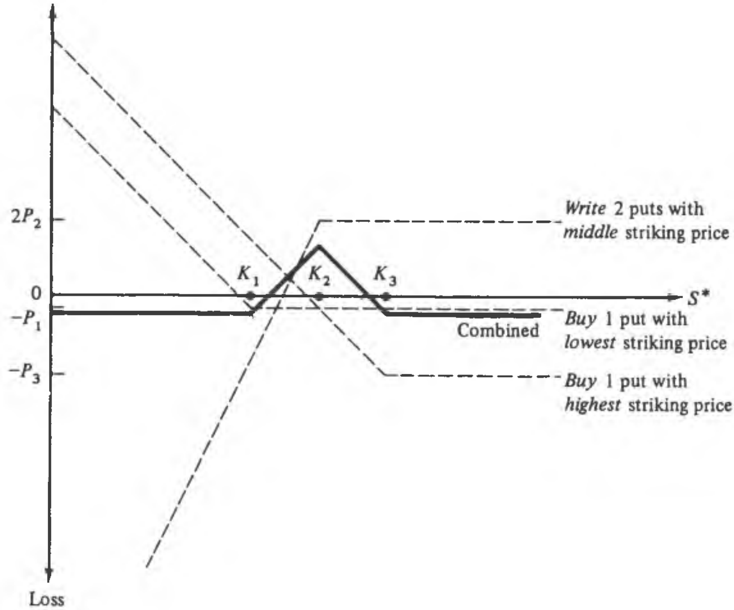


Figure 1-18 Butterfly Spread ($S = K_2$)

A *combination* combines options of different types on the same underlying stock so that they are either both bought or both written.

The most popular combination combines a put and a call on the same underlying stock, with the same striking price and the same expiration date. This is termed a *straddle*. An example is shown in Table 1-3. If written at-the-money, this straddle profits only if the stock price remains near the common striking price. In contrast to a butterfly vertical spread with calls, where the middle calls are written, this straddle has greater potential for profit and loss, while the butterfly has less maximum profit and a limited loss. In a written vertical combination around-the-money, the written call is out-of-the-money, and the written put is in-the-money. While having the same range of profit as a straddle, the profit triangle is flattened.

As Figures 1-19 through 1-24 suggest, combinations have *bottom* and *top* versions, depending on whether the options are bought or written. This same terminology may be applied to the more complex forms of hedges in Figures 1-8, 1-9, 1-11, and 1-12. In general, top positions are described by

Table 1-3
STRADDLE

Striking Price	*Type*	*Expiration Month* *JAN*	*APR*	*JUL*	*Stock Price*
XYZ 30	*C*	11	$12\frac{3}{8}$	$13\frac{1}{2}$	40
XYZ 30	*P*	$\frac{7}{16}$	$1\frac{1}{8}$	$1\frac{3}{4}$	40
XYZ 40	*C*	$4\frac{1}{2}$	$6\frac{1}{4}$	$7\frac{7}{8}$	40
XYZ 40	*P*	$3\frac{1}{2}$	$4\frac{3}{4}$	$5\frac{1}{2}$	40
XYZ 50	*C*	$1\frac{1}{4}$	$2\frac{7}{8}$	$4\frac{3}{8}$	40
XYZ 50	*P*	$10\frac{1}{4}$	11	$11\frac{1}{2}$	40

upward pointing triangles, and bottom positions by downward pointing triangles. "Top" indicates a maximum profit limit, and "bottom" a maximum loss limit.

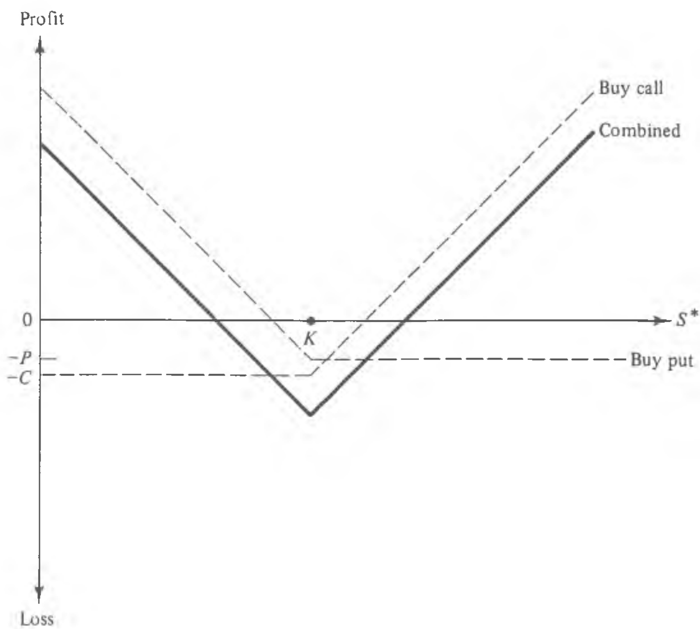


Figure 1-19 Bottom Straddle ($S = K$)

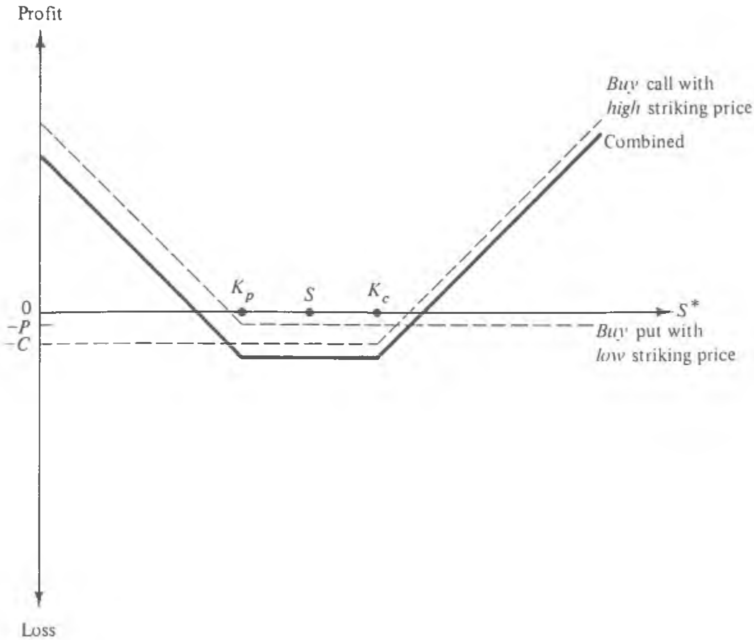


Figure 1-20 Bottom Vertical Combination

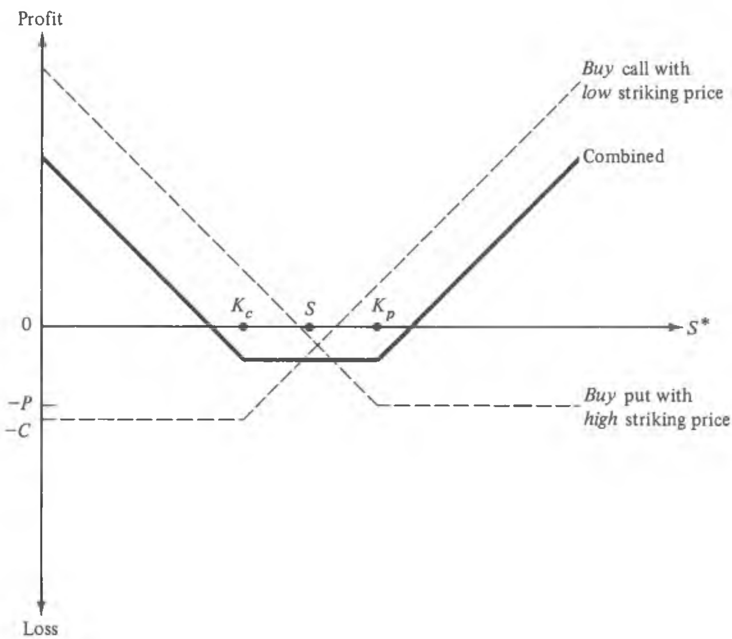


Figure 1-21 Bottom Vertical Combination

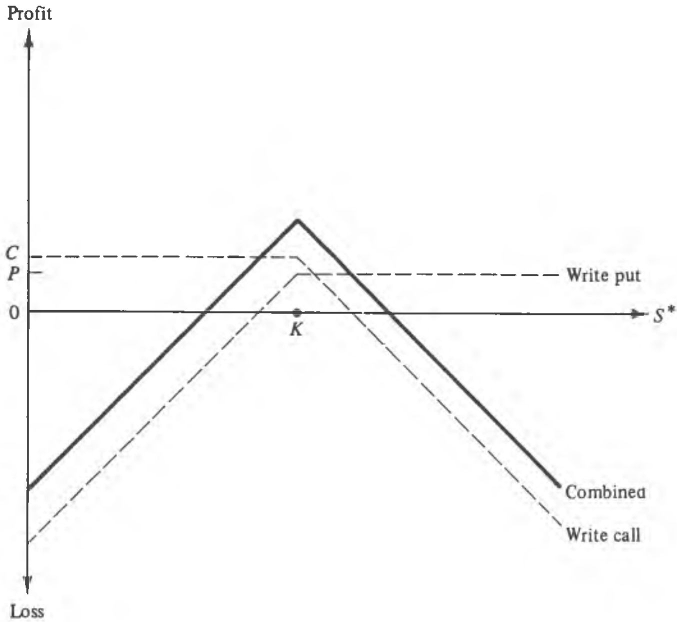


Figure 1-22 Top Straddle ($S = K$)

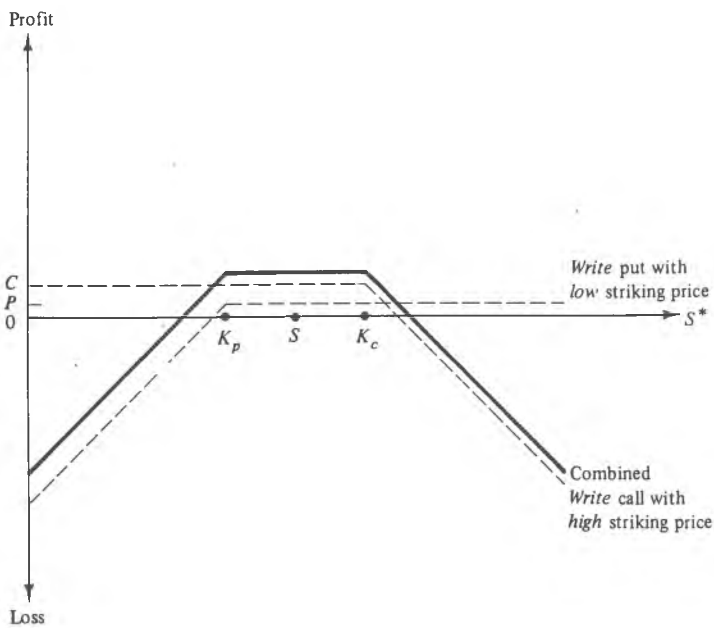


Figure 1-23 Top Vertical Combination

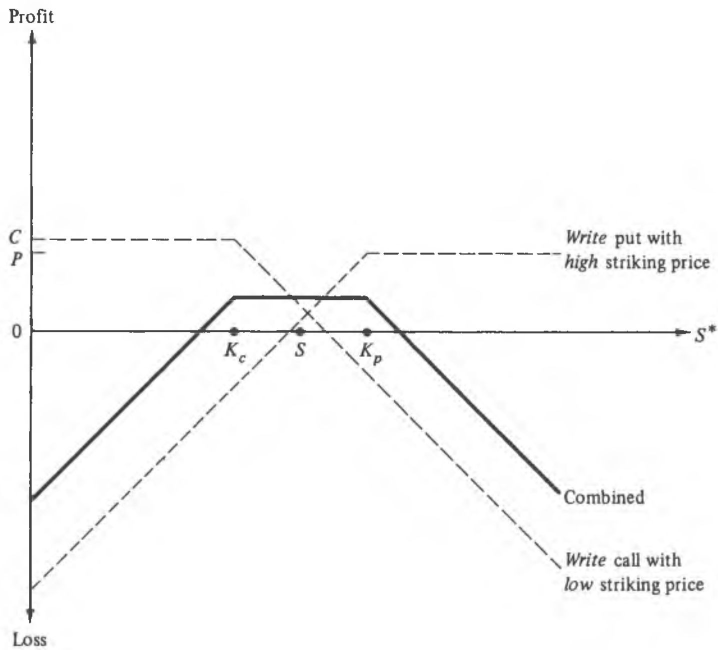


Figure 1-24 Top Vertical Combination

These payoff diagrams provide a good way to become familiar with various investment strategies. However, to interpret them correctly it is important to remember their limitations. They are valid only if all parts of the position are held until expiration. Since most of them involve selling as well as buying options, one could not be sure that they would be held until expiration unless the options could be exercised only at that time. In this case, the options are popularly termed *European*, in contrast to their *American* counterparts, which can be exercised at any time on or before the expiration date.[6] Although these labels do have some historical justification, nearly all options now traded in Europe, as well as in the United States, are of the American type. Nevertheless, we will often find it useful to consider European options.

A few additional qualifications should be kept in mind. If a strategy involves holding a long or short position in the stock, as in Figures 1-7 through 1-12, then the corresponding diagram is valid only if the stock does not pay a dividend during the life of the position. One could informally adjust for this by adding in an amount for dividends received by the stock,

[6] A European option obviously cannot be worth more than an otherwise identical American option. However, this does not imply that it will always be worth less. We will discuss this more fully in Chapter 4.

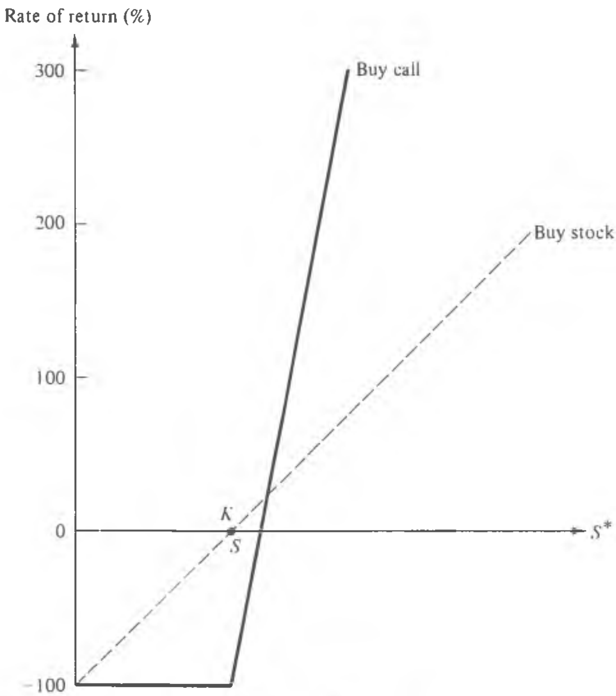


Figure 1-25 Rate of Return Diagram

or subtracting it if the stock is sold short. Also, the vertical placement of each diagram, and the corresponding profit or loss, of course depends on the initial price specified for the options.[7] An alternative that avoids this dependence is the final value diagram, which gives the value of the position at the expiration date for each possible final stock price. Finally, the profit and loss shown do not include the time value of the money invested in the position. The investment must be made now, but the final value is not received until the expiration date. For some purposes it might be helpful to adjust for this by calculating profit and loss relative to the amount to which the initial investment would have grown if it had been invested in default-free bonds.

Another way to illustrate the implications of options positions requiring a positive net investment is with a rate of return diagram. In this case, the profit on the vertical axis is replaced by the corresponding rate of return earned on the net investment in the position. Figure 1-25 compares the rate of return on a long position in one share of stock with the rate of

[7] Figures 1-1 through 1-24 correspond to the July option prices shown in Tables 1-1, 1-2, and 1-3.

return on a purchased call option. It shows the considerable difference in risk of equal dollar investments in stock and options.

1-3. A BRIEF SURVEY OF THE WAY OPTIONS ARE TRADED

In the United States, puts and calls have had a history of sporadic acceptability since their first appearance in 1790. The popular misconception equating options with gambling has resulted in extensive government regulation, with puts and calls at times considered illegal. The Securities Act of 1934 empowered the Securities and Exchange Commission (SEC) to regulate options trading, and the Put and Call Brokers and Dealers Association was formed to represent option dealers. Although very small during the 1940s, options volume increased considerably during the next two decades. However, even by 1968, annual contract volume reached only 302,860, representing about 1% of New York Stock Exchange (NYSE) volume, measured in terms of share equivalents.

In retrospect, it is easy to see why the over-the-counter (OTC) market fared so poorly. First, transactions costs were very high. Purchase or sale of an *OTC call* generated three types of direct transactions costs: endorsement fees (to guarantee performance of an option in the event of exercise), the dealer spread, and, typically, a further commission paid by the buyer. Indirect transactions costs resulted from subsequent equity sales or purchases in the event of exercise.

For example, on a call contract (rights to 100 shares) for which the public writer received $200, the public buyer would typically have paid $250. The $50 difference was split among the writer's share of the endorsement fee ($12.50), the buyer's share of the endorsement fee ($12.50), and the dealer bid-ask spread ($25). In addition, the buyer usually paid a $6.25 commission, so the call would actually cost him $256.25. In the event of exercise, the writer of a call would pay a regular stock commission on the forced sale of stock to the call buyer. If he had not already purchased the stock, he would pay another commission. Finally, after receiving payment for the stock sold due to exercise, the writer would usually reinvest the proceeds, generating another commission.

Transactions costs on *OTC puts* were even greater. Since the supply of written puts was usually larger than the demand from put buyers, an intermediary, called a "converter," would buy the put from the writer, and through a series of arbitrage transactions in the underlying stock and default-free instruments, would transform the purchased put into a call, for which there was the requisite demand. Just how this was done need not concern us now; it will become apparent later in Chapter 2. It was, however, a costly process, adding approximately $20 to the total transactions cost. In effect, the difference in the price paid and received for a put contract would typically be $70, not including the buyer's commission.

A second reason for the low volume of the OTC options market was that no convenient secondary market was available for puts and calls. Buyers and writers were essentially committed to their positions until the expiration date. Not only did this force equity commissions on options finishing in-the-money, but also considerably increased a writer's exposure to risk.

The recent and most significant change in the options market was a response to these and other deficiencies in the OTC market. On April 26, 1973, the Chicago Board Options Exchange (CBOE) became the first registered exchange for trading listed call contracts. At the time, a total of 48 option series were traded to 16 underlying securities for three different maturities. Initially, 305 seats were sold for $10,000 each. Average daily contract volume during May, the first full month of operation, equaled 1,584, with an average of 1.7 contracts per trade. In a short time, interest in listed options trading has far surpassed initial projections. The CBOE, given its short tenure, has been the most successful securities exchange in the history of U.S. capital markets. By March 1974, monthly contract volume on the CBOE exceeded the entire 1972 annual volume of the earlier over-the-counter market. By the end of 1974, in terms of share equivalents, volume exceeded shares traded on the American Stock Exchange. In December 1983, calls and puts on 145 underlying securities were listed on the CBOE for an average daily contract volume of 364,977, with an average of 13.5 contracts per trade. In many cases, daily volume in underlying stocks was typically exceeded by daily volume in their associated options. CBOE open interest, the number of outstanding option contracts at any point in time, had grown 421 times, from 16,222 at the end of May 1973 to 6,840,625 at the end of December 1983. Membership in the CBOE had expanded to 1,753 by December 1983, with the last sale seat price for the year of $212,000.

Since the opening of the CBOE, calls have been listed on the American Stock Exchange (AMEX) commencing January 13, 1975, on the Philadelphia Stock Exchange (PHLX) commencing June 25, 1975, on the Pacific Stock Exchange (PSE) commencing April 9, 1976, and on the Midwest Stock Exchange (MSE) commencing December 10, 1976.[8] Listed put option trading was initiated on all five exchanges on June 3, 1977. For the year 1983, average daily option contract volume totaled across all exchanges equaled 536,201. The market shares for the year were 52.8% for the CBOE, 26.7% for the AMEX, 12.3% for the PHLX, 8.2% for the PSE. Under-

[8] The options market of the MSE was combined with the CBOE on June 2, 1980.

standably, OTC contract volume has fallen off, and many OTC dealers have stopped trading. The remaining interest in the OTC market derives principally from longer-than-one-year option maturities on underlying stocks with listed options and options on active and volatile stocks that fail to meet listing requirements.

The success of the organized options exchanges can be attributed to several innovations.

1. The creation of a *central marketplace*, with its attendant regulatory, surveillance, and price-dissemination capabilities. The exchanges distribute pamphlets on trading in options (on tax aspects or spreading strategies, for example) and make their annual reports and the prospectus of the Options Clearing Corporation available to the public.[9]

2. The introduction of the *Options Clearing Corporation* as the single guarantor of every CBOE, AMEX, PHLX, and PSE option affords greater protection to option buyers than arrangements provided by the over-the-counter market. The buyer of a contract looks directly to the Clearing Corporation, and not to any particular writer, for performance in the event of exercise.

3. In creating a *secondary market*, the Clearing Corporation stands as the opposite party to every trade, making it possible for buyers and sellers to terminate their positions at any time by an offsetting transaction. Prior to the CBOE, buyers and writers of options in the over-the-counter market were essentially committed to their positions until the expiration date.

4. On the CBOE and PSE (but not on the AMEX or PHLX), the *broker/dealer functions of the specialist are separated.* For example, on the CBOE, the broker function is handled by *Floor Brokers* and *Order Book Officials*, who may not trade for their own account; and the dealer function is performed by *Market Makers*, who may only trade for their own account. Public orders are therefore filled exclusively by Floor Brokers and Order Book Officials, while Market Makers, trading from their own capital and inventory, provide liquidity to the market. To reduce further the potential for conflict of interest and to increase liquidity, the CBOE, as well as the PSE, has instituted a competitive Market Maker system. With the exception of certain limited memberships, any Market Maker can trade in any option at any time. In addition, for each underlying security, a number of

[9] Materials prepared by the AMEX may be ordered from the Publications Department, American Stock Exchange, 86 Trinity Place, New York, New York 10006. CBOE publications can be obtained by writing to the Marketing Services Coordinator, The Chicago Board Options Exchange, LaSalle at Jackson, Chicago, Illinois 60604. The CBOE also provides a seminar kit for brokers certified to deal in options and makes available on loan a film describing trading on the exchange floor.

Market Makers are assigned the responsibility of trading in the options on a regular basis.

5. Unlike the over-the-counter markets, the exchanges employ *certificateless trading*. The ownership of an option is evidenced by confirmations and monthly statements received by customers from their brokers. This facilitates one-day business settlement on option sales and purchases and reduces costs.

6. *Standardization* of the terms of option contracts has also served to decrease transaction costs.

Options on the same underlying stock have either a January/April/July/October *expiration cycle*, a February/May/August/November cycle, or a March/June/September/December cycle. At any time, options are available with the three expiration dates of the cycle nearest to the present. Therefore, the longest maturity an option can have is nine months. In its expiration month, an option expires at 10 : 59 P.M. Central Time (11 : 59 P.M. Eastern Time) on the Saturday immediately following the third Friday of the month.[10] This is the final time an option can be tendered to the Clearing Corporation. However, secondary market trading of the option ceases at 3 : 00 P.M. Central Time on the business day immediately preceding the expiration date. Furthermore, to exercise an option, a customer must instruct his broker no later than 4 : 30 P.M. Central Time on the business day immediately preceding the expiration date.[11] The calendar in Table 1-4, supplied by the CBOE, marks the expiration dates for 1981 and the Quotron Option Retrieval Code.

Striking prices are chosen by the exchanges from among those permitted by the SEC. As of 1984, the allowable prices were integers evenly divisible by 5, in 5-point intervals for striking prices up to $100 and in 10-point intervals for striking prices over $100. Striking prices that violate this rule are allowed if they arise from adjustments for stock splits or stock dividends, as described below. If an exchange chooses to do so, it may use larger intervals for at least some stocks, and thus omit some possible striking prices. For example, an exchange could choose to use 5-point intervals for striking prices up to $50, 10-point intervals for striking prices between $50 and $200, and 20-point intervals for striking prices over $200.[12] When options with a new expiration date are to be introduced, an exchange will

[10] Prior to July 1974, listed options expired on the last business day of their expiration month. From July 1974 to January 1976, the expiration date was the last Monday of the expiration month.

[11] Each brokerage firm has its own rules regarding the deadline for receiving exercise instructions, which may be prior to the 4 : 30 deadline set by the exchanges.

[12] These were the striking price intervals that were allowed by the SEC prior to October 31, 1980.

Table 1-4
EXPIRATION CALENDAR

OE Expiration Calendar—1981

ARY, 1981
T W T F S
1 2 3
6 7 8 9 10
13 14 15 16 17
20 21 22 23 24
27 28 29 30 31

APRIL, 1981
S M T W T F S
1 2 3 4
5 6 7 8 9 10 11
12 13 14 15 16 17 18
19 20 21 22 23 24 25
26 27 28 29 30

JULY, 1981
S M T W T F S
1 2 3 4
5 6 7 8 9 10 11
12 13 14 15 16 17 18
19 20 21 22 23 24 25
26 27 28 29 30 31

OCTOBER, 1981
S M T W T F S
1 2 3
4 5 6 7 8 9 10
11 12 13 14 15 16 17
18 19 20 21 22 23 24
25 26 27 28 29 30 31

JANUARY, 1982
S M T W T F S
1 2
3 4 5 6 7 8 9
10 11 12 13 14 15 16
17 18 19 20 21 22 23
24 25 26 27 28 29 30
31

APRIL, 1982
S M T W T F S
1 2 3
4 5 6 7 8 9 10
11 12 13 14 15 16 17
28 19 20 21 22 23 24
25 26 27 28 29 30

JULY, 1982
S M T W T F S
1 2 3
4 5 6 7 8 9 10
11 12 13 14 15 16 17
18 19 20 21 22 23 24
25 26 27 28 29 30 31

THE FOLLOWING CBOE UNDERLYING STOCKS EXPIRE DURING THE ABOVE CYCLE

A ● ARC BS FNC DIS ● XON ● FLR ● HM ● IBM IP MRK MTC PEP SOB ● TDY WY
XP ● AVP ● BNI DAL ● DD ● FDX GWF ● HOI ● HR JNJ ● MER NWA ● PRD ● STK ● TXN ● XRX
BAM BGH DEC ● EK FNM ● HAL INA IGL ● KMG MMM PZL SY TAN UPJ

JARY, 1981
T W T F S
3 4 5 6 7
10 11 12 13 14
17 18 19 20 21
24 25 26 27 28

MAY, 1981
S M T W T F S
1 2
3 4 5 6 7 8 9
10 11 12 13 14 15 16
17 18 19 20 21 22 23
24 25 26 27 28 29 30
31

AUGUST, 1981
S M T W T F S
1
2 3 4 5 6 7 8
9 10 11 12 13 14 15
16 17 18 19 20 21 22
23 24 25 26 27 28 29
30 31

NOVEMBER, 1981
S M T W T F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
29 30

FEBRUARY, 1982
S M T W T F S
1 2 3 4 5 6
7 8 9 10 11 12 13
14 15 16 17 18 19 20
21 22 23 24 25 26 27
28

MAY, 1982
S M T W T F S
1
2 3 4 5 6 7 8
9 10 11 12 13 14 15
16 17 18 19 20 21 22
23 24 25 26 27 28 29
30 31

AUGUST, 1982
S M T W T F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31

THE FOLLOWING CBOE UNDERLYING STOCKS EXPIRE DURING THE ABOVE CYCLE

AEP ● BLY ● BA CEA CWE GF ● HIA ● JM ● NSM RJR SO UAL JWC
AHS BAX BCC KO ● CDA HRS ● HON MGI ● OXY ● SLB ● SN UNC ● WCI
AMP BDK CBS CL ● GD ● HWP IFF ● MOB RTN SKY TG UTX ● WMB

I, 1981
T W T F S
3 4 5 6 7
0 11 12 13 14
7 18 19 20 21
4 25 26 27 28
1

JUNE, 1981
S M T W T F S
1 2 3 4 5 6
7 8 9 10 11 12 13
14 15 16 17 18 19 20
21 22 23 24 25 26 27
28 29 30

SEPTEMBER, 1981
S M T W T F S
1 2 3 4 5
6 7 8 9 10 11 12
13 14 15 16 17 18 19
20 21 22 23 24 25 26
27 28 29 30

DECEMBER, 1981
S M T W T F S
1 2 3 4 5
6 7 8 9 10 11 12
13 14 15 16 17 18 19
20 21 22 23 24 25 26
27 28 29 30 31

MARCH, 1982
S M T W T F S
1 2 3 4 5 6
7 8 9 10 11 12 13
14 15 16 17 18 19 20
21 22 23 24 25 26 27
28 29 30 31

JUNE, 1982
S M T W T F S
1 2 3 4 5
6 7 8 9 10 11 12
13 14 15 16 17 18 19
20 21 22 23 24 25 26
27 28 29 30

SEPTEMBER, 1982
S M T W T F S
1 2 3 4
5 6 7 8 9 10 11
12 13 14 15 16 17 18
19 20 21 22 23 24 25
26 27 28 29 30

THE FOLLOWING CBOE UNDERLYING STOCKS EXPIRE DURING THE ABOVE CYCLE

APA CHA ● GLW EVY ● GE ● HT ● KN MSU ● OI ● REV S ● TEK
● BMY CGP ● DOW F ● GM ITT ● LIT ● NCR ● RAL ● ROK ● SOC
BC CSC ● ESM ● FT ● GW KM MCD ● NWT RCA SA ● SYN

OPTION PRICE RETRIEVAL CODES

To retrieve option prices, first key in stock symbol. Then add appropriate expiration month code (put or call) and striking price code

iration Month Codes

nths	Calls	Puts	Month	Calls	Puts	Month	Calls	Puts
uary	A	M	February	B	N	March	C	O
l	D	P	May	E	Q	June	F	R
	G	S	August	H	T	September	I	U
ober	J	V	November	K	W	December	L	X

Striking Price Codes

Striking Prices			Code	Striking Prices			Code	Striking Prices			Code
5	105	205	A	35	135	235	G	70	170	270	N
10	110	210	B	40	140	240	H	75	175	275	O
15	115	215	C	45	145	245	I	80	180	280	P
20	120	220	D	50	150	250	J	85	185	285	Q
25	125	225	E	55	155	255	K	90	190	290	R
30	130	230	F	60	160	260	L	95	195	295	S
				65	165	265	M	100	200	300	T

imples: Alcoa Jan/50/Call—AAAJ
General Motors April/70/Call—GMDN
Control Data Feb/60/Put—CDANL
Hewlett Packard Nov/80/Put—HWPWP

○ Expiring option classes cease trading today
◇ Expiration date
□ Exchange holiday
● Puts traded

NOTE: New nine-month options are ordinarily introduced on the first business day (usually Monday) following an expiration date.

CBOE EXPIRATION CALENDAR published with the permission of the Chicago Board of Options Exchange.

usually select, for the interval size chosen, the two striking prices closest to the current stock price. If the stock price is very close to one of these striking prices, then often that striking price and the surrounding two will be selected. If the stock price subsequently reaches or moves beyond the highest or lowest existing striking price, then ordinarily trading will be opened two days later in a new striking price, the end point of the new interval. However, without special permission from the SEC, an option cannot be opened with less than 45 days remaining until its expiration or while the underlying stock trades below $6.

With the exception of certain adjustments, described below, *contract units* of options are in "round lots" of 100; that is, one option contract represents rights to 100 shares of the underlying stock. Unlike OTC options, *no adjustments are made for cash dividends.*[13] However, adjustments are made for stock splits and stock dividends by proportionately increasing the number of shares of the underlying stock covered by the option and by decreasing the striking price. For example, consider a single option covering 100 shares of stock with a striking price of $50 per share. Suppose that the stock splits 5 for 4 or, equivalently, issues a 25% stock dividend. Then, after the adjustment, the option will cover 125 ($100 \times \frac{5}{4}$) shares of stock with a striking price of $40 ($\$50 \times \frac{4}{5}$) per share. However, if a stock split is 1 for 1 or more whole shares, the number of shares covered by an option is not adjusted. Instead, the number of outstanding options is proportionately increased and the striking price is proportionately decreased. Thus if the split were 2 for 1 rather than 5 for 4, after the adjustment the single option would be replaced by two options, each covering 100 shares and having a striking price of $25 per share. In any case, at the time of the adjustment, additional new options with standard terms are usually introduced. These new options have contract units of 100 shares and striking prices chosen according to the rules given in the previous paragraph. Consequently, options that differ in contract units but are identical in every other way may sometimes be traded simultaneously. In the event of recapitalizations, reorganizations, or other distributions, the Clearing Corporation will attempt to adjust the terms of outstanding options in a way that will be fair to both buyers and sellers.

The *underlying securities* chosen for listing must meet a number of requirements. They must be registered and listed on a national securities exchange; be widely held (at least 7,000,000 outstanding shares held by at least 6,000 shareholders); meet a minimum trading volume requirement

[13] For this purpose, cash dividends are specified as cash distributions from "earnings and profits" as defined in the Federal Internal Revenue Code. Note that a call buyer would have to exercise the call on the business day *just prior* to the ex-dividend date or earlier to receive the dividend. On the other hand, a put buyer who also holds the underlying stock must wait until the ex-dividend date or after to exercise the put and still receive the dividend.

(2,400,000 shares per year); have a closing market price of at least $10 per share for the prior three-month period; have a record of not defaulting on sinking fund installments, interest, principal and preferred dividend payments during the prior three years; and have a minimum after-tax net income of $1,000,000 for the prior eight quarters. Securities can be voluntarily withdrawn from listing by any exchange, or they are mandatorily withdrawn if they fail to meet certain similar but less stringent requirements. If a security is withdrawn, options trading will continue in all outstanding striking prices and expiration dates, but no new options will be introduced. If a delisted security subsequently satisfies the requirements, it can be listed again. Many underlying securities that would be attractive for listing in terms of their anticipated option trading volume fail to meet one or more of the requirements. It has not been uncommon for an exchange to request permission of the Securities and Exchange Commission to list a new underlying security within a short time after it first qualifies for listing. At the time of this writing, all underlying securities are traded on either the NYSE or the AMEX.

Table 1-5 illustrates how option price information is presented in the *Wall Street Journal.* The particular figures shown are an excerpt from the quotations for December 6, 1983. Each row contains the prices for all options on a given stock with a given striking price. Along each row, the call and put prices are shown for each expiration month. The prices are quoted on a per-share basis, so that the cost of an option contract is given by multiplying the option price by the number of shares covered. Options trading under $3, in terms of rights to one share, trade in sixteenths of a point, while those over $3 trade in eighths.

The recorded prices for options represent the last trade of the day. The closing stock prices reported in the *Wall Street Journal* on the options page may differ from the corresponding stock prices reported on the stock pages. Since January 26, 1976, the high, low, and close reported on the stock pages are composite prices drawn from a pooled ticker tape from all U.S. exchanges. In particular, for NYSE stocks dually listed on the Pacific Stock Exchange, which remains open after the NYSE close, the composite close is likely to represent the last trade on the PSE. However, to value an option at any point in time, we need to know the simultaneous stock price. Since all U.S. options exchanges, as of this writing, close with the NYSE, the NYSE closing stock price is more relevant to value an option at the close of trading than is the composite price.[14] Consequently, the *Wall Street Journal* is careful to record the NYSE closing stock price on the options page. Unfortunately, if all the option quotes for a particular stock

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[14] To allow a more orderly closing, during 1979 the CBOE, MSE, and PSE initiated the practice of closing at 3 : 10 P.M. Central Time, ten minutes after the NYSE close.

Table 1-5

OPTIONS QUOTATIONS FROM THE *WALL STREET JOURNAL**

Philadelphia Exchange

Option & NY Close	Strike Price	Calls–Last Dec	Calls–Last Mar	Calls–Last Jun	Puts–Last Dec	Puts–Last Mar	Puts–Last Jun
AirPd	40	r	6¼	r	r	r	r
46⅛	45	1⅛	3¼	r	½	2¼	r
46⅛	50	1-16	1⅛	r	r	r	r
AnheusB	60	4⅝	6⅜	r	1-16	1	1¾
64⅜	65	⅝	3½	5¼	1	2½	3¼
64⅜	70	⅛	1⅝	3½	r	5⅞	6½
64⅜	75	r	½	r	r	10½	r
ArchDM	20	2⅛	r	r	r	r	r
22	25	r	½	1⅛	r	r	r
Arch o	23¾	1-16	1¼	s	r	r	s
Avco	30	4¼	5¼	r	r	9-16	¾
34⅜	35	¼	r	r	1⅛	r	r
BecDic	35	2¼	r	4⅛	¼	⅞	r
36¾	40	⅛	1½	2	3½	3¾	4¼
36¾	45	r	¼	1	r	r	8¼
36¾	50	r	r	s	13¾	r	s
Chartr	10	3-16	15-16	1 5-16	½	1⅜	1⅝
9¾	15	r	3-16	5-16	r	r	r
ChFC o	26⅝	2	s	s	r	s	s
ChFChk	25	3⅝	5⅛	r	1-16	⅞	r
28½	30	½	2 11-16	3½	1¾	3	r
28½	35	⅛	1	1⅞	6¾	r	r
DonLJ	15	1¼	r	r	⅛	¾	r
16⅛	20	r	⅝	1⅛	r	r	r
EGG	35	¾	2⅜	3½	⅞	2⅛	r
34⅝	40	r	⅝	r	r	r	r
E D S	37½	1	s	s	1¼	s	s
36½	30	r	r	r	1-16	¾	r
36½	35	2⅜	5	7	r	2⅝	3½
36½	40	⅜	2¾	4½	3⅜	5¼	r
36½	45	1-16	1⅝	3¼	r	r	9½
Emery	20	r	7	r	r	½	r
26¼	25	r	3¾	4¼	½	1¾	r
26¼	30	s	1¼	2¼	s	r	r
FinC o	20	r	3	s	⅜	r	s
20⅞	23⅜	⅛	1¾	r	2⅝	3⅝	r
20⅞	26⅝	r	⅞	r	5⅞	6⅛	r
20⅞	30	1-16	r	1⅜	9⅜	r	r
FinCp	20	1⅛	3¼	4¼	3-16	2	r
20⅞	25	⅛	1½	2⅝	r	5	r
GnInst	30	2⅜	4⅜	5¾	¼	1 9-16	r
32⅛	35	¼	2¼	3⅜	2¾	4¼	5⅛
32⅛	40	1-16	1 1-16	2	8	8⅜	r
32⅛	45	r	⅝	s	r	r	s

Option & NY Close	Strike Price	Calls–Last Dec	Calls–Last Mar	Calls–Last Jun	Puts–Last Dec	Puts–Last Mar	Puts–Last Jun
Getty	55	22½	s	s	r	s	s
77¾	60	17⅝	18⅛	r	r	½	r
77¾	65	12⅞	r	15⅝	1-16	1	1⅝
77¾	70	7¾	10⅜	11¼	⅛	1¾	3⅛
77¾	75	3⅝	7½	r	13-16	3½	r
77¾	80	s	5	7⅜	s	6¼	7
LearSg	40	6¼	6¾	r	r	r	r
46⅝	45	2	3½	5¼	½	r	r
46⅝	50	3-16	1¼	2¼	3¼	r	r
MartnM	35	r	3¼	r	r	1¼	r
36¾	40	¼	1	2¼	r	r	r
MrtM o	40	3-16	r	r	r	r	r
36¾	43⅜	1-16	r	r	r	r	r
Morgan	65	r	7⅛	r	r	1	r
70¾	70	1⅛	r	r	¾	r	3½
70¾	75	1-16	1⅜	r	4¾	r	6
NMedCr	15	¼	1¼	1⅝	r	r	r
Newmnt	45	9⅜	r	11¼	r	r	r
54⅛	50	4⅛	r	r	r	r	r
54⅛	55	¾	3½	r	1⅛	r	r
54⅛	60	1-16	r	r	r	r	r
OwensC	35	1¼	2⅞	r	r	r	2⅝
36	40	r	r	1¾	4¼	r	r
PennC	30	9	9⅝	s	r	r	s
38⅞	35	3⅞	r	6½	r	r	r
38⅞	40	⅜	2⅛	r	r	2⅜	r
38⅞	45	r	¾	r	r	r	r
TexOG	40	r	r	r	r	1⅛	r
43¼	45	⅜	2⅞	r	r	3	r
43¼	50	r	1 5-16	r	6⅝	6⅜	r
43¼	55	1-16	r	r	r	r	r
Textrn	35	r	1½	r	1⅜	r	r
33⅞	40	r	⅜	r	r	r	r
Time	60	3¼	5½	r	⅛	2⅜	r
63⅝	65	¾	3½	r	2¾	3¾	r
63⅝	70	1-16	r	r	r	r	r
TriCtl	25	3⅞	r	r	r	r	r
28⅝	30	1-16	½	r	r	r	r
WmsEl	10	⅛	13-16	1	1⅛	1⅜	1⅞
9⅛	15	r	3-16	7-16	r	5¾	r
9⅛	20	1-16	r	s	r	r	s

are for the same striking price, then everything is listed on one line and no NYSE closing stock price is given. For example, no stock price is quoted for National Medical Care in Table 1-5. In such cases, only the composite close from the stock pages will be available.

The letter "r" means a particular option is available for trading but did not trade during that day. The letter "s" means no option of the corresponding type, underlying stock, expiration date, and striking price has been opened by the exchange. Moreover, if no options for a given underlying stock and striking price have traded during the day, no reference to them will appear in the listing, even though at least some of them are available for trading. For example, on this particular day, there were no trades in any Avco options with a striking price of 40. Consequently, they are not mentioned in Table 1-5, even though options were available for trading in all three expiration months.

The letter "o" indicates certain options that have nonstandard contractual terms as a result of a stock split, stock dividend, spinoff, or special circumstances. This designation is customarily used in situations where the identities of some of the options might otherwise be unclear. The options on Martin Marietta provide an example. Martin Marietta stock split 3 for 2 on October 18, 1983. Subsequently, the outstanding options were adjusted according to the rules given above; each old contract now covered 150 shares, with a striking price equal to $\frac{2}{3}$ of the former amount. For the options whose original striking price was $60, the adjusted striking price was $40; these are the options listed as MrtM o 40. The quotes designated as MartnM are for the new options with standard contract sizes that were introduced after the split. The contracts of the same type and expiration date listed as MartnM 40 and MrtM o 40 are thus identical except that one covers 100 shares and the other covers 150 shares. Their last trades, however, could have occurred at quite different times during the day. Consequently, it is not surprising that Table 1-5 shows different prices for the two December 40 calls.

Usually no special identification is given when the adjustment for a stock split or stock dividend results in a standard contract size. In that case, adjusted and new options with the same striking price would be identical in every way and no distinction would be needed. For example, this was the situation when Electronic Data Systems split 2 for 1 on June 8, 1983, and no special designation for its options appears in Table 1-5. Also, normally no special identification is given when the terms of an adjustment are such that none of the adjusted options has a standard price. Occasionally, a stock with a special designation from a split will split a second time during the life of a listed option. In that event, the original options, which will then have been adjusted twice, are denoted by the symbol "oo," while the options introduced after the first split but before the second will be indicated with an "o."

Mergers, acquisitions, and spinoffs often result in the options of one company being adjusted to cover a certain number of shares of one or more other companies. In such instances, the *Wall Street Journal* policy is to give no quotation for the closing stock price, even when all of the securities concerned are publicly traded. On some occasions, however, the *Journal* has followed the confusing practice of quoting the closing price of only one of the securities in the package underlying the option.

The *Wall Street Journal* also gives, for each exchange, total volume and open interest figures for both puts and calls. The total volume figures tell the number of contracts traded during the day.[15] The open interest figures show the total number of contracts outstanding as of the end of the day. Unfortunately, the *Wall Street Journal* gives separate volume figures for each option only for the most actively traded options. However, *Barron's*, a financial weekly, does quote the weekly volume and open interest at the end of the week for most individual options. *Barron's* also gives the high, low, and closing prices for the week for each option.

The brief institutional description of organized option markets provided in this section is supplemented by Chapter 3, which describes in considerable detail the placement of a public order to buy or sell an option, the role of the Options Clearing Corporation, floor trading procedures, margin requirements, listed option commissions, and tax aspects of option trading.

[15] These figures are the *reported* volume for the day. For a number of reasons they may differ from *cleared* volume, which is the actual number of contracts for which payment is subsequently made. For example, reported trades may be cancelled due to a misunderstanding between the buying and selling parties that is not reconciled until the next day. More frequently, cleared volume exceeds reported volume since some trades are not reported, particularly during very active trading periods. However, as a general rule, reported and cleared volume will not be significantly different.

some fundamental aspects of options

2

2-1. DETERMINANTS OF OPTION VALUE

In Chapter 1, we looked at the values of various options positions on the expiration date. At that time, an option's value depended on only two variables—the stock price and the striking price. However, at any time before expiration, a number of other variables will also be important. In fact, the differences among the option prices that we see in Table 1-5 are the result of the interaction of a number of different forces.

The only way we can hope to sort out their effects is to take them one at a time, so that is what we will do. We will always make the following comparison: If two options are alike in every way except for a single variable, how will their values differ? If we find some difference, we will call that variable a direct determinant of option value. If we find no effect, then we will have to conclude that if this variable influences option values at all, it must do so *indirectly* through its effect on the direct determinants.

The same set of variables matters for both puts and calls, but not always in the same way, so we will talk about calls first and then come back to puts. Here is our list of the six fundamental direct determinants of option value:

1. **Current stock price (S)**
2. **Striking price (K)**

3. **Time to expiration (t)**
4. **Stock volatility**
5. **Interest rates**
6. **Cash dividends**

The first candidates for the list are easy to agree upon. The *stock price* S and the *striking price* K will certainly matter before expiration as well as on the expiration date. The higher the stock price, the higher the call value. Similarly, the higher the striking price, the lower the call value.

Another important determinant is the stock's *volatility*. For the moment, we can think of the volatility as a measure of the dispersion of possible future stock prices.[1] The higher the volatility, the greater the likelihood that the stock will do either very well or very poorly. These are offsetting effects for the owner of the stock, but not for the owner of a call. He will get the full dollar benefit from the favorable outcomes, but will avoid most of the dollar loss from the unfavorable outcomes, since in those cases he will not exercise the call. Consequently, the higher the volatility over the lifetime of a call, the higher is its value relative to the stock.[2]

If time is measured in years, then *time to expiration* t measures the fraction of a year remaining in the life of the option. For currently listed puts and calls, t is $\frac{3}{4}$ (that is, nine months) or less. One effect of a longer time to expiration works in the same way as a higher volatility. Over a long period of time, a lot can happen to even a very low volatility stock. The call premium over parity therefore tends to be higher the more time remaining

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[1] Volatility is not the same thing as a stock's "beta." Volatility is a measure of the variability of the stock price, while beta is a measure of the stock's sensitivity to overall market movements. Beta is thus a measure of the component of risk which cannot be diversified away (nondiversifiable risk). Empirically, high volatilities and high betas often go together, but not always. Gold stocks, for example, are very volatile, but they tend to have low betas. For a discussion of beta coefficients, and modern portfolio theory in general, see William F. Sharpe, *Investments*, 2nd Ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1981).

[2] Note that volatility is a measure of the total risk of a stock. If you have studied modern portfolio theory, this may seem puzzling. You have probably heard again and again: Total risk is irrelevant, only nondiversifiable risk matters. It turns out that option valuation and portfolio theory are completely consistent, but it will not be clear why this is so until much later in the book.

In the meantime, here is an example that will show that if we measure volatility by nondiversifiable rather than total risk, we will get results that are obviously wrong. Suppose that we wish to value two options with identical terms on two different companies. The current stock price of each company is the same and is below the present value of the striking price. Suppose also that interest rates are constant. The first company invests only in government bonds; it has neither nondiversifiable risk nor total risk. The second company also has no nondiversifiable risk (a zero beta), but it has substantial total risk. The option on the first company has a zero value, since with certainty the stock price will never be above the striking price. If volatility were measured by nondiversifiable risk, the option on the second company would also have zero value. But this is ridiculous, since there is a positive probability that it will finish in-the-money.

before expiration. Since this premium shrinks to zero as the expiration date approaches, a call is sometimes interpreted as a "wasting asset."

The higher the *interest rate*, the lower the present value of the striking price the call buyer has contracted to pay in the event of exercise. From this effect, a higher interest rate will have the same influence as a lower striking price.[3] Consequently, higher interest rates tend to imply higher call values.[4]

The present value of the striking price decreases as t increases, so time to expiration has a second way of influencing the call value: first, by providing more time for changes to occur in the stock price, and second, through its effect via the rate of interest. For a call, these effects reinforce each other.

One other fundamental determinant of call values remains. Listed options are protected against stock splits and stock dividends. However, unlike over-the-counter options, they provide no protection for *cash dividends.*[5] This affects the values of listed calls in *two* ways. To understand why, we must first analyze how cash dividends affect the price of shares.

To a first approximation, the average stock price change on an ex-dividend date will be lower than the average change on other days by the amount of the dividend.[6] For any given current stock price, higher future dividends come at the expense of lower future price appreciation. In the extreme case, a final liquidating dividend would drive the stock price to zero. Ordinary dividends can be interpreted as partial liquidation of the firm with a resulting lowering of the stock price. To examine this more carefully, let S be the stock price just before the stock goes ex-dividend and S^x be its expected price just after. Let D be the amount of the dividend. Suppose $S^x > S - D$, so that the stock price falls by less than the dividend. Leaving aside taxes, margin requirements, and transactions costs, a good strategy would be to buy the stock just before it goes ex-dividend, then sell it just after the ex-dividend date. Since we are then entitled to the dividend, we pay S and receive $S^x + D$.[7] Since we assumed $S^x + D > S$, we will expect

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[3] At first it might seem that the present value of both the cost of exercising *and its benefit* (receiving the stock) will be lower, so that the overall effect is ambiguous. But the present value of a random future sum is simply the amount you would have to pay today to secure the ownership of that sum. No matter what the interest rate, to secure today the ownership of a random future stock price, you have to pay the current stock price, S. But we are changing only one variable at a time, so S, and therefore the present value of the benefit, remains constant.

[4] The full influence of interest rates will also depend on, among other things, their own volatility and their correlation with the stock price.

[5] With over-the-counter options, the striking price is typically reduced by the amount of the dividend on each ex-dividend date. This provides partial but not complete payout protection.

[6] If you sell a stock prior to its ex-dividend date, you transfer the right to receive the dividend to the buyer. If, instead, you sell a stock *on or after* its ex-dividend date, then you retain the right to receive the dividend.

[7] To avoid trivial details, we will assume throughout the book that cash dividends are received on the ex-dividend date.

to earn a profit. If this were not possible, we must instead require $S^x \leq S - D$. A similar argument shows that if $S^x < S - D$, we could short the stock, then buy it back and expect to earn a profit.[8] Therefore, if such strategies permitting expected profit are not available, we must expect $S^x = S - D$: The stock price is expected to fall by the amount of the dividend.[9]

The owners of the stock get both components of total return, cash dividends and price changes. The holders of unprotected calls can receive no benefit from cash dividends, only from price changes. Hence, it stands to reason that the larger the fraction of total return made up by cash dividends to be paid *with ex-dividend dates prior to the expiration date*, the lower the call value. The second influence of cash dividends is related to the first, but is more complex. It concerns the optimal timing of exercise, and will be treated later in Chapter 4. The basic idea is very intuitive, however. The time remaining in the life of the option now has a third effect, which works in the opposite direction from the previous two: The longer the time to expiration, the more the stock price will be reduced by cash dividends. If this effect becomes dominant, it will be advantageous for the owner of a call to end its life voluntarily by exercising it.

These same six factors listed on pp. 33 and 34 also influence put values. Other things equal, puts should be more valuable the lower the stock price, the higher the striking price, and the lower the interest rate—just the reverse of a call. However, increased stock volatility, by raising the probability of extreme outcomes, increases both put and call values.

These effects are straightforward, but the influence of dividends merits some discussion. A long position in a put can be compared to a short position in a stock. Anyone who is short the stock does not benefit from price declines due to a dividend being paid, because he or she is required to make restitution for the dividend to the person from whom the stock was borrowed. On the other hand, the owner of a put does get the full benefit of price declines due to dividends as well as price declines due to other factors. Hence, as dividends increase, a put becomes more attractive relative to the stock itself as a vehicle for going short. The higher the cash dividends prior to expiration, the higher the value of a put. In addition, as explained in Chapter 4, cash dividends will affect the optimal exercise strategy for puts as well as calls.

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[8] Of course, at best we could actually only initiate the position at the close of trading on the business day just prior to the ex-dividend date and close out the position at the opening on the ex-dividend date. In the interim, we would be exposed to other factors besides the dividend affecting the stock price, but, with so little time between the transactions, this risk would be quite minor. In addition, the transactions costs to market professionals are very small.

[9] As an institutional practice, open orders for stock are automatically reduced by the dividend on the ex-dividend date.

The only surprise is the influence of time to expiration. Even with no dividends, this variable has separate contrary effects: Greater time to expiration tends to decrease put values by reducing the present value of the net proceeds from exercise of the put, and it tends to increase put values by widening the dispersion of possible future stock prices. At low stock prices (relative to K), the former effect dominates, since increased dispersion then has a relatively small influence on put values. Moreover, exercise at or before expiration is likely, and the present value of the net proceeds is more sensitive to the time of their receipt. For opposite reasons, the latter effect dominates at high stock prices. In brief, if it were not for the possibility of early exercise, at a sufficiently low current stock price, put values would decrease with a longer time to expiration, and at a higher stock price, put values would increase with a longer time to expiration. However, as shown in Chapter 4, the possibility of early exercise insures that put values never decrease when time to expiration increases.

Table 2-1 summarizes our analysis of option values to this point.

Table 2-1
SOME DETERMINANTS OF OPTION VALUE

	Effect of Increase	
Determining Factors	*P*	*C*
1. Current stock price (S)	↓	↑
2. Striking price (K)	↑	↓
3. Time to expiration (t)	↑	↑
4. Stock volatility	↑	↑
5. Interest rates	↓	↑
6. Cash dividends	↑	↓

NOTE: An arrow pointing in one direction means that the effect cannot be in the opposite direction.

Are there any other significant factors that can affect option values? There are, but they differ in essential ways from the ones we have just listed. Those six variables are fundamental determinants of option values. They will always matter. The following four additional variables appear to be equally basic. In some circumstances they would be, but not always. We will find, most surprisingly, that in many very important situations they have no direct influence at all:

7. **Expected rate of growth of the stock price**
8. **Additional properties of stock price movements**
9. **Investors' attitudes toward risk**
10. **Characteristics of other assets**

The expected rate of growth of the stock price would seem to be one of the most obvious determinants of option value. Since call values are higher the greater the current stock price, intuition would strongly suggest that, *other things equal*, the current value of a call would be higher the greater the stock price is expected to be in the future. On the other hand, the same intuition implies that put values, *other things equal*, should be lower the greater the expected growth rate of the stock price. However, with the development of the put-call parity relationship in the next section, it will be easy to see this intuition must be incorrect! As incredible as it may seem, the expected rate of growth of the stock price may not be a direct determinant of option value.

The importance of additional properties of stock price movements really depends on what is included in the volatility variable. If we allow it to be a multidimensional measure, then it is possible for it to encompass all of the relevant information about the probability distribution of future stock prices, so further variables may be unnecessary.

Modern financial theory suggests that investors' attitudes toward risk and the characteristics of other assets should play a critical role in determining the value of any asset, including an option. However, this does not mean that these variables must be direct determinants of option value. Indeed, they may affect option values only indirectly, through their influence on the stock price, stock volatility, and interest rates.

In summary, the seventh through tenth variables are all ones that could conceivably influence option values. When they do, their effects will be relatively complex. Fortunately, many results of great practical value can be obtained without having to deal with these complications.

Four more variables, related to the institutional environment, may also affect option values. These are:

11. **Tax rules**
12. **Margin requirements**
13. **Transactions costs**
14. **Market structure**

In principle, these variables will always matter, but their significance may be small. Chapter 3 describes the current structure of options markets and discusses prevailing tax rules, margin requirements and transactions costs. A detailed analysis of the effects on option values of every facet of these rules and regulations would be very tedious, since they are so involved, and unrewarding, since they frequently change. Instead, in the subsequent chapters, we will show general ways for examining the effects of these variables.

We do not mean to imply that the market price of options will never be affected by yet other additional factors. Rather, we are saying that these other factors should influence option values only through their effect on the variables we have mentioned. If they have a direct influence, then market prices will differ from underlying values, and this may provide especially attractive investment opportunities.

In this section, we have discussed option valuation only on an informal basis. This discussion serves as an introduction to Chapters 4 and 5, which present a considerably more detailed and precise treatment of option valuation. In Chapter 4, under the assumption that no riskless profitable arbitrage opportunities are available in the options market, we develop general properties which the values of puts and calls must possess. In Chapter 5, by additionally characterizing the path of the stock price as it moves through time, we derive an *exact* formula relating the value of a given put or call to the six fundamental factors: the current stock price, the striking price, time to expiration, stock volatility, interest rates, and cash dividends.

2-2. THE RELATIONSHIP BETWEEN PUTS AND CALLS

An important and surprising relationship exists between the values of puts and calls with the same expiration date and striking price and written on the same underlying stock. We are introducing this relationship now for several reasons. Comparison of Figures 1-5 and 1-7 alerts us to the fact that a strategy of buying the stock, writing a call, and buying a corresponding put will produce a constant profit (or loss), no matter what the final stock price turns out to be. Surely this implies something about put and call values. Furthermore, the relationship between puts and calls provides a good illustration of some parts of our discussion in the previous section. We could pursue these questions using payoff diagrams. Instead, as a prelude to Chapter 4, we will use an *arbitrage table*. This table describes the returns of a specially constructed portfolio of securities associated with the same underlying stock. The future value of the portfolio is calculated for each possible level of the stock price at the expiration date. By applying the simple principle that a portfolio yielding zero returns in every possible situation must have zero current value to prevent riskless profitable arbitrage, we can derive the relationship between put and call values.

To focus on the basic issues, we assume that there are no transactions costs, margin requirements, or taxes. We believe that for examining arbitrage relationships, these simplifications are as appropriate as they are convenient. In the long run, we would expect recurring arbitrage opportunities to be whittled away to the point where there is no longer a profit to even the most advantageously situated traders. These persons will undoubtedly be market professionals with extremely low transactions costs and margin requirements. Also, if all sources of income are taxed at the same rate, which approximates the situation for a professional trader, then the relationship remains unchanged when taxes are included. We also assume that it is possible to borrow and lend at the same rate. Again, that is not unreasonable in this context. The main reasons private borrowing rates exceed lending rates are transactions costs and differences in default risk. Transactions costs per dollar decline rapidly as the scale increases, so they are of secondary importance in a large operation. And if the arbitrage operation in which we are using the borrowed funds is indeed riskless, it should be possible to collateralize the loan so that the lender will bear no possibility of default. Furthermore, interested readers can easily modify the relationship between put and call values to include all of the things we have left out.

Before proceeding, we need to introduce one more symbol related to interest rates. We define r^{-t} to be the number of dollars that would have to be paid today in order to obtain one dollar with certainty at time t from now. Thus, in return for the loan of one dollar now, we will receive r^t dollars at time t from now. In other words, r^{-t} is a *present value* factor. If the payment of the striking price K will be made at time t from now, then Kr^{-t} is its present value. In terms of bond prices, r^{-t} dollars is the current price of a default-free bond[10] with time t until maturity, paying one dollar on its maturity date and nothing before then; in general, r will depend on the time to maturity. If t is measured in years, then for each maturity date the corresponding annualized interest rate is $r - 1$. For example, suppose that the current price of a bond paying \$1 three months from now is \$0.96. Then an investment of \$1 now will give \$1/.96 = \$1.042 at the end of three months. If we could reinvest the principal and accumulated interest on the same terms every three months for a year, then we would have $\$(1.042)^4 = \$1.1789$ at the end of the year for each \$1 initially invested. Hence, we can refer to 17.89% as the annualized interest rate corresponding to the quarterly rate of 4.2%. (Sometimes the corresponding annualized rate is quoted as simply four times the quarterly rate, 4(4.2%) = 16.8%, but this understates the true return because it ignores the fact that the accumulated interest can be reinvested.)

Table 2-2 reviews all of the symbols and their definitions.

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[10] Strictly speaking, no bond can be default-free. Here we refer to bonds whose value is negligibly affected by the possibility of default. For example, we regard bonds issued by the U.S. government as "default-free." By the same token, any arbitrage operation can at best be virtually riskless. For example, there is always the remote possibility of government action nullifying some contracts.

Table 2-2
SYMBOLS

S	= current market price of underlying stock
C	= current value of an associated call
P	= current value of an associated put
K	= striking price
S^*	= market price of underlying stock on expiration date
t	= time to expiration
r	= one plus the rate of interest on a default-free loan over a given period

We will first consider European puts and calls on a stock which will pay no dividends during the life of the options. All results derived assuming no dividends will also hold with dividends if the options are *payout-protected*. An option is payout-protected if its contractual terms are adjusted in a way that will make its value insensitive to cash dividends. Subsequently, we will see what happens for unprotected options on stocks that do pay cash dividends.

Consider taking the following simultaneous position in a *European put and call* on the same underlying stock with the same striking price K and time to maturity t: write one call, buy one put, buy one share of stock, and borrow Kr^{-t} by selling zero-coupon bonds with time t to maturity. As shown in Table 2-3, this gives you the amount $C - P - S + Kr^{-t}$ now. On the expiration date, if $S^* \leq K$, the put you bought will be worth $K - S^*$ and the call you wrote will expire worthless. On the other hand, if $S^* > K$, then the call will be worth $S^* - K$, and you will let your put expire unexercised. In either case, you will own the stock, worth S^*, and will owe K to

Table 2-3
ARBITRAGE TABLE ILLUSTRATING PUT-CALL PARITY RELATIONSHIP FOR PAYOUT-PROTECTED EUROPEAN OPTIONS

	Current Date	Expiration Date	
		$S^* \leq K$	$K < S^*$
Write call	C	—	$K - S^*$
Buy put	$-P$	$K - S^*$	—
Buy stock	$-S$	S^*	S^*
Borrow	Kr^{-t}	$-K$	$-K$
Total		—	—

repay your borrowing. Hence, the future cash flow will be zero in all possible circumstances. The reverse position obtained by buying one call, writing one put, shorting one share of stock, and lending will also give a future cash flow of zero in all possible circumstances. Consequently, if there are to be no arbitrage opportunities, it must be true that the initial investment required to set up either of these positions is also zero:

$$C - P - S + Kr^{-t} = 0,$$

which can be rewritten as

$$\boxed{C = P + S - Kr^{-t}.}$$

This equation is known as the *put-call parity relationship* for European options on stocks that pay no dividends.

If this relationship were violated in actual markets and we ignored transaction costs, margin, and taxes, we could make a certain profit on zero investment by selling the relatively overpriced option and using the proceeds to buy the relatively underpriced option, together with an appropriate position in the stock and borrowing or lending. The remaining proceeds would be our sure profit, since the portfolio would require no cash outflow (or inflow) on the expiration date of the options.

To see how this might work, suppose we can invest in two four-month ($t = \frac{1}{3}$) options, both with striking price $K = \$40$. Their underlying stock price is $S = \$40$, and the annualized interest rate on a four-month loan is 5% ($r = 1.05$). Suppose, further, the put and call were available at \$2 and \$3, respectively. Since $Kr^{-t} = \$39.35$, with the call selling at \$3, we know from the put-call parity relationship that the put should be worth \$2.35. Since the put is then underpriced relative to the call, we can be sure of a profit if we

- Write one call at \$3
- Buy one put at \$2
- Buy one share at \$40
- Borrow \$39.35 at 5% annual rate to be paid back in four months.

This nets us \$0.35 immediately, representing the extent of relative underpricing of the put. On the expiration date, no further gain or loss results. To see this, if the stock price remains at \$40, both the put and call expire unexercised, and proceeds from the sale of the stock will exactly retire the loan. If the put finishes in-the-money, the call would expire unexercised, we can deliver the stock upon exercise of the put, and the \$40 proceeds from exercise will exactly retire the loan. If, instead, the call finishes in-the-money, the put would expire unexercised, we can deliver the stock when the call is exercised against us, and the \$40 proceeds from exercise will exactly retire the loan. In any event, no net cash outflow is required at the expiration date. We therefore net an immediate profit of \$0.35. Had we expanded the position to 10 option contracts for 100 shares each on both sides, our profit would have been \$350.[11]

What effect do cash dividends have on the put-call parity relationship? If we can predict the cash dividends with certainty prior to expiration of an option, we can make an exact correction in this relationship. As we concluded earlier, on its ex-dividend date, a stock has a tendency to experience a decline in price roughly equal to the amount of the cash dividend per share. This causes puts to be worth more and calls to be worth less than their payout-protected values. Let D be the sum of the present values of all cash dividends to be paid with *ex-dividend dates prior to the expiration date* of an option. The European put-call parity relationship for unprotected options then becomes

$$C = P + S - D - Kr^{-t}.$$

In other words, the stock price is replaced by the difference between itself and the present value of the dividends. The reader should assure himself or herself that violation of this relationship presents an opportunity for profitable riskless arbitrage. In Chapter 4, we will expand this relationship to allow for uncertain cash dividends.

To summarize, for European puts and calls, knowing only

1. The put price
2. The underlying stock price
3. Interest rates
4. Anticipated cash dividends

fully determines the value of an associated call with the same expiration date and striking price as the put. This implies there is *basically only one type of European option*—the put; all other positions (calls, spreads, combinations) can be created by an investor on his own account. Moreover, regarding the put as an insurance contract against the risk of a long position in stock, a call is a combination of insurance and a levered long position in the stock. Alternatively, of course, we can regard the call as basic and the put as derived from its associated call.

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[11] In this example, the borrowed funds are simply a bookkeeping item which permits us to realize immediately a profit equal to the underpricing of the put. They are unnecessary to the arbitrage transaction. Without them, we would show a certain rate of return on our investment greater than the interest rate.

The formula, with certain dividends, can also be rearranged as $S = C - P + D + Kr^{-t}$, implying that a long position in the stock can be replicated by a portfolio containing a purchased call, a written put, and lending of the amount $D + Kr^{-t}$.

Finally, we can use the formula to shed some light on an issue raised in Section 2-1: the effect on option values, other things equal, of the expected rate of growth of the underlying stock price. The put-call parity relationship says that $C = P + S - D - Kr^{-t}$. On the left side we have the call value; on the right side we have the put value plus five of the fundamental determinants of option value: S, K, D, r, and t. This means that if we change *any other variable* that may affect option value while holding these five constant, then the change must affect put and call values in exactly the same way. For example, an increase in volatility must increase both the put and call values by the same amount. This squares with our intuition. Likewise, if a higher expected rate of growth of the stock price, other things equal, were to increase the call value, then it must also increase the put value. Since this contradicts our intuitive arguments in Section 2-1, our intuition must be wrong.

At first, the explanation seems to be that we have made a comparison that could never occur. But this is a red herring; such a situation could definitely happen.[12] We must conclude that the expected rate of growth affects option values in a more subtle manner than initially contemplated. This in turn makes it much easier to believe that the expected rate of growth may have no direct influence at all on option values. As a practical matter, we believe this is at least very close to being the case. It is not a logical necessity, however. In Section 7-1 we give an example where an increase in the expected rate of growth increases both put and call values.

Finally, we want to emphasize once again that all parity relationships developed thus far apply to *European options* only. Listed options are American. In Chapter 4, these relationships will be extended to encompass American puts and calls.

2-3. WHY INVESTORS USE OPTIONS

There are many reasons why investors may find options useful. Some of them are obvious, but others will require some reading in subsequent chapters to be fully appreciated. Here we can give only a broad overview.

[12] Here is an example. Modern portfolio theory suggests that for a given dividend policy the equilibrium expected rate of growth of a stock will be determined by its degree of nondiversifiable risk. Even the theory's severest critics admit that there must be some truth to this. But nondiversifiable risk is only one component of volatility. So it is perfectly possible for two stocks to have the same current value, the same dividend policy, the same volatility, but different degrees of nondiversifiable risk and hence different expected rates of growth.

We will not try to provide a general framework for making investment decisions. Such an attempt would take us far off track, and there are many other books offering this information. We will instead take it for granted that everyone interested in options is making decisions in a careful, intelligent way, has considered both his or her long-run objectives and the range of available alternatives, and wants no further general advice. Our goal is to show that options can offer investors a somewhat wider range of alternatives than they may have realized. The question we wish to ask is thus, What can options do for you that stocks and bonds cannot? However, we cannot resist quoting one of the most important messages of modern work on investments, mainly because it is so relevant to many of the reasons we give for using options. The message is not surprising, for it is really just good common sense: *Diversification offers many advantages; in a highly competitive environment, it is very difficult to obtain information not already reflected in market prices; consequently, one should be very cautious about giving up diversification in an attempt to use special information, especially when significant transactions costs will be incurred as well.*

Some of the additional opportunities provided by options will exist only in certain circumstances. Many of them may be valid for some investors but not for others. Nevertheless, the rapid growth of the exchange-traded options indicates that many investors have found that at least one of the following reasons applies to them.

1. OPTIONS MAY OFFER A PATTERN OF RETURNS THAT COULD NOT BE OBTAINED WITH THE STOCK. This is probably the most often mentioned reason for using options. After all, Figures 1-1 and 1-2 look quite different from Figures 1-3 and 1-4. However, this comparison can be very misleading because it takes into account only *fixed buy-and-hold* positions in stock and default-free bonds. It overlooks the possibilities provided by dynamic strategies that make subsequent adjustments in the amount of stock and bonds held. To find out what options offer over and above stock and bonds, we will have to examine these additional opportunities very carefully.

With this in mind, suppose we set out to find a dynamic strategy for a stock and bond portfolio which will make it as much like a call (on one share of stock) as possible. A brief study of daily or weekly call prices will show that they tend to have the following features:

1. Call prices and stock prices change in the same direction.
2. A \$1 change in the stock price causes a change of less than \$1 in the call price.
3. A 1% change in the stock price causes a change of more than 1% in the call price.

Hence, at a minimum we would want the value of our stock and bond portfolio to have these same three features. This will be very easy. To satisfy the first two conditions, we will want to have a long position in less than one share of stock. We can meet the third condition by financing our stock position partly through borrowing (that is, by selling bonds). For example, suppose that we buy $\frac{1}{2}$ share of a stock selling for $100 per share by investing $10 and borrowing $40. The current value of our portfolio is $10—$50 worth of stock minus the $40 owed on the borrowing. If the stock price now goes up by $1, then the value of our portfolio will go up by only $0.50. However, this will be a 5% increase for the portfolio, compared to only 1% for the stock.

Some further observation of call prices will show that they have additional important properties which we will want to match. When a call is deep-out-of-the-money, a $1 change in the stock price has little effect on the call price. If the stock price rises so that the call becomes at-the-money, then as a rough rule of thumb, a $1 change in the stock price produces a $0.50 change in the call price. If the stock rises further so that the call becomes deep-in-the-money, then a $1 move in the stock price produces nearly a $1 change in the call price.

Here is where the opportunity to use a dynamic strategy becomes essential. We want to be holding almost no shares when the stock price is low, and we want to be buying more shares as the stock price rises. In particular, when the call is at-the-money, we want to be holding about $\frac{1}{2}$ share. As the stock price rises further and the call becomes deep-in-the-money, we want to have gradually bought in to the point where we are now holding almost one share. Similarly, whenever the stock price falls, we will want to reduce the number of shares held.

We will also want the adjustments we are making to depend on the passage of time. On the expiration date, we want to hold one share of stock if the call is in-the-money or no shares at all if it is out-of-the-money. We could do this by gradually increasing the number of shares we hold at any given stock price as time passes for stock prices greater than the striking price, and gradually reducing the number of shares held at each stock price less than the striking price.

There is one more very important property we would want the portfolio to have. After a call is purchased, no subsequent out-of-pocket expenditures are ever required, nor are any funds ever received until the position is closed out. The same should be true for our portfolio. To meet this condition, we will finance new purchases of stock by selling more bonds (that is, by borrowing more), and we will use any proceeds from the sale of stock or from dividends to buy bonds (that is, to repay part of the borrowing).

Strategies having all these features will obviously get much closer to the returns of a call than any buy-and-hold strategy. But could we find one which would duplicate a call *exactly*? In some circumstances of great interest, the answer turns out to be yes. In Chapter 5, we will show precisely how this can be done and examine its profound implication for option pricing. You may have already guessed what that will be. If we can indeed find some dynamic stock and bond portfolio which will require no subsequent investment and will be worth exactly $\max[0, S^* - K]$ on the expiration date, then the current value of that portfolio must be the fair value of the call.[13]

Furthermore, if we can duplicate a call, then we can duplicate any other type of option position as well. Table 2-4 shows some of the properties of the corresponding equivalent portfolios for several basic positions.

Table 2-4

STOCK-BOND PORTFOLIOS EQUIVALENT TO OPTIONS

		Long Stock (less than one share)		Short Stock (less than one share)			
		+ Long bonds (lending)	+ Short bonds (borrowing)	+ Long bonds (lending)	+ Short bonds (borrowing)		
As Stock Price Rises	Buy stock and sell bonds	Long stock (one share) + Long one put	Long one call	Long one put	Short stock (one share) + Long one call	Sell stock and buy bonds	As Stock Price Falls
	Sell stock and buy bonds	Long stock (one share) + Short one call	Short one put	Short one call	Short stock (one share) + Short one put	Buy stock and sell bonds	

For each option position, the corresponding stock and bond portfolio is given at the top of its column and the appropriate revision strategy is given at the ends of its row. For example, the table says that buying a put is equivalent to a short position in the stock combined with lending, which

[13] In that case, it is really proper to say that call prices have the features we have listed *because* the equivalent portfolio has them, rather than the other way around.

will be revised by buying more bonds and shorting more stock when the stock price falls and by selling back bonds and buying back stock to reduce the short position as the stock price rises.

Can we then conclude that we can always duplicate a call using only stock and bonds and that options can therefore never offer a new and different pattern of returns? No, this is not the case at all. To see what could go wrong with a duplication strategy, let us return to Section 2-1. Our arguments there imply that an unanticipated increase in volatility will increase the value of a call. It is certainly conceivable that such a change could occur without affecting the price of the underlying stock (or of bonds). Consequently, the value of the call would change but the value of our portfolio would not, no matter what dynamic strategy we were using. An unanticipated change in a firm's dividend policy would cause the same problem. Once again, the value of the call could change without any corresponding change in the value of a stock and bond portfolio.

One further circumstance could also derail an attempt to duplicate a call exactly. It involves the possibility of a sudden large jump in the stock price.[14] Here is an example. Suppose a company has as its only assets a group of copper mines in a foreign country. The price of the company's stock will normally fluctuate with the price of copper and general economic conditions. However, there is a small, but continual, probability that a coup will occur. If this happens, the mines will be nationalized and the stock will be worthless. A coup could succeed only if it were completely secret, so there will be no advance warning and the news will be available to many people simultaneously. In such a catastrophe, stop-loss orders or portfolio revision strategies will be of no help in limiting losses—there will be no buyers at any price. Now a call could provide something that a levered position in the stock cannot—a way to insure that the losses do not exceed the value of the call.

It seems that in some circumstances we will be able to duplicate an option and in other circumstances we will not. Which of these situations is of practical relevance? In our opinion, they both are. Certainly, we are unlikely to find a stock which will never have an unanticipated change in volatility or dividend policy and will never make a sudden jump. However, it is also unlikely that we will find a stock for which these factors are so important that we cannot construct a portfolio which will be very similar to an option. For this reason, we strongly feel that the concept of an option being equivalent to a carefully adjusted portfolio of stock and bonds is close enough to being true in most situations of practical interest to make it an

[14] When we say that the stock price makes no sudden jumps, we mean that even though the stock price may move very quickly from, say, 50 to 51, it will still be possible, if we wish to do so, to execute trades at $50\frac{1}{8}$, $50\frac{1}{4}$, $50\frac{3}{8}$, and so on.

invaluable tool for understanding options. Accordingly, we will use it as the context for explaining our next four reasons for using options. On the other hand, we feel that these factors are important enough for us to conclude that options can in fact give a pattern of returns that could not be obtained with stock and bonds. Indeed, it is the possible significance of these factors that leads to our sixth, seventh, and eighth reasons for using options.

Furthermore, even when there is a duplicating strategy, it will typically involve a considerable amount of trading. Thus it may happen that a market professional with very low transactions costs can effectively duplicate an option by trading in stock and bonds, while it would be very impractical for an individual investor to do so. For the individual investor, the option and the equivalent portfolio of stock and bonds may offer identical returns before transactions costs are included, but not afterward. If such an individual would in fact like to have a portfolio that is continually readjusted in a way equivalent to some option position, then he would be better off achieving it indirectly but automatically with the option rather than directly with stock and bonds.

In any case, the automatic readjustment feature of options would still not make them useful if there were no investors who would like to change their mix of stocks and bonds as stock prices change. But it is probably self-evident that many people would like to do this. Two categories immediately come to mind. First, there are those whose degree of risk aversion changes as their wealth changes. Some people prefer to reduce their total exposure to risk when their wealth decreases. Similarly, when their wealth increases, they feel that they can then afford to take more chances. Others react in just the opposite way. When their wealth increases, their inclination is to protect their higher standard of living by taking less risk. Accordingly, when their wealth decreases, they would be willing to accept more risk to try to recoup their losses. Although all of this could be done directly by adjusting a portfolio of stocks and bonds, each group might benefit from having a way to make the desired adjustments automatically. Options can provide this. A portfolio of calls and bonds would exhibit the behavior wanted by the first group. A portfolio of stock and written calls would meet the requirements of the second group. Option funds now make it easier for individual investors to hold diversified portfolios of options. However, in either case, an investor would ideally like to use options on his total portfolio, or perhaps on a market index, for this purpose. A portfolio of options would be the next best thing, but, as we will see in Chapter 8, it is not the same as an option on a portfolio.

In the second category are those who feel that the sequence of past price movements conveys information about future price movements. For example, an investor may have some rather questionable information indicating that a stock is a good buy. A common way of reacting to this is as follows: If the stock moves upward, this is a good sign that the information was right, and consequently I would like to increase my holdings; if the stock goes down, then I will assume that it was a mistake and reduce my holdings. A call will make the neçessary adjustments automatically. Of course, many more examples like this could be given, and in each case the desired portfolio revision strategy could often be accomplished directly with the stock. But usually an investor will also have available as an alternative an option position which will provide the same pattern of returns automatically.

2. Options may offer you an opportunity to borrow or lend at more favorable rates than you can obtain elsewhere. We have just discussed how in many situations options may be equivalent to a portfolio containing a long or short position in a stock and some amount of borrowing or lending. But at what interest rates are this borrowing and lending implicit in options being done? The answer is, usually at the rates available to large market participants. For many individuals, these rates will be more favorable than they can obtain on their own. They may thus find it advantageous to borrow or lend indirectly in options markets rather than combining stock positions with direct borrowing or lending.

3. Options may in effect allow you to take a position in a stock under more favorable margin restrictions than would be available directly in the stock market. Margin requirements really involve three separate things: limits on borrowing against long positions, limits on the use of the proceeds from short sales, and requirements for collateral to guarantee performance on short sales. Current regulations limit the amount of borrowing that can be done using the stock as collateral to 50% of the stock's value; no borrowing at all is allowed against options. The remaining two requirements concern short sales. A short sale of borrowed stock generates funds equal to the price of the stock. Who gets the use of the money? Large investors are often able to negotiate a rental fee directly with the lender of the stock. In effect, the short seller and the lender divide the use of the money. Small investors are less fortunate. They have little choice but to arrange their short sales through brokerage firms, which typically use stock held in street name[15] to make the sale. Neither the short seller nor the actual owner of the loaned stock receive the use of any of the money. The funds are kept by the brokerage firm and the interest they earn is at least partly passed along to all customers in the form of reduced charges. Furthermore, an individual will typically have to put up more funds, in addition to those generated by the short sale, as a performance bond to

[15] When an individual finances a stock purchase partly by a loan from a brokerage firm, the stock must be left with the firm and registered in its name; that is, the stock is held in street name.

guarantee his ability to cover the short sale. But now there is an important difference. The individual can receive the interest on these additional funds, so this requirement imposes no economic loss on individuals with sufficient capital. In summary, more favorable margin requirements are always desirable. Any investor would benefit from having the use of the funds from a short sale, and investors with insufficient capital may also benefit from less stringent borrowing limits and collateral requirements. Although all margin requirements fall into the broad categories given here, their actual computation can be very complicated, particularly with combined positions. We will discuss these calculations in much more detail in an appendix to Chapter 3.

Options may allow individuals to obtain more favorable margin requirements than would be available directly in the stock market. Again, the easiest way to see this is to consider the situations where an option is equivalent to a portfolio of stock and bonds. We have seen that a put is then equivalent to a portfolio combining a short position in the stock with lending. The total amount loaned can be broken into two parts: an amount equal to the value of the stock sold short and an amount equal to the value of the put. Brokerage firms and certain other institutions would have no difficulty accomplishing this; they have the full use of the proceeds from a short sale. However, if an individual sets up such a portfolio directly, margin requirements would prevent him from receiving the proceeds of the short sale and lending them. He would only receive the interest from the second component of the loan, which was equal to the value of the put. By taking a position in the options market instead, he may be able to obtain much more favorable margin requirements. In effect, he may be able to obtain full use of the proceeds from the short sale of stock implicit in the purchase of a put.

A similar argument applies to written calls. They would be equivalent to a short position in the stock combined with lending, but now the total amount of lending is less than the value of the stock sold short. The difference is the value of the call. So we could think of this as a short sale equal to the amount of the lending, plus another short sale equal to the amount of the call. Margin regulations normally allow the writer to receive the use of the proceeds from the sale of the call. In effect, he too has received complete use of the proceeds from the short sale of stock implicit in the sale of a call. Of course, if the written call is uncovered, some collateral will also be required, but even this may be less than would be necessary with a direct short sale of stock.

So far we have looked at options positions that give a short position in the stock—buying puts and writing calls. Long positions—buying calls or writing puts—may also offer margin advantages. Margin requirements limit the amount of borrowing that can be done using the stock as collateral to a fixed percentage of the stock's value. We have seen that a call may be equivalent to a long position in the stock combined with borrowing. For some calls, especially those out-of-the-money, this implicit borrowing is a much higher percentage of the stock's value than would be allowed if the position were taken directly in the stock market. This is true even though no borrowing is allowed on the purchase of the call. For individuals wanting a highly levered position in a stock, the options market may offer the best, or even only, way of obtaining it.

The writer of a put may be able to obtain a long position in the stock with even greater leverage. We have said that buying a put is equivalent to a short position in the stock combined with lending an amount greater than the value of the short sale. So the sale of a put is equivalent to buying the stock by borrowing its entire value, and then borrowing some more. The amount of this additional borrowing is equal to the value of the put. If you are an uncovered writer, some collateral will be necessary. This requirement can be met with interest-bearing securities, so it will cause no actual loss of interest. However, it will reduce the borrowing potential. In effect, you will be forced to loan back part of the borrowing. But the remaining borrowing may still be more than you could obtain directly in the stock market.

4. OPTIONS MAY OFFER TAX ADVANTAGES UNAVAILABLE WITH STOCK AND BONDS. No one has ever accused our tax laws of being too simple. Indeed, we will need to devote an appendix to Chapter 3 just to a survey of the parts concerning options. Nevertheless, we can get some idea of the effects of taxes without going into specific details. Anyone who is completely unfamiliar with the tax laws may wish to read this appendix before reading the next few paragraphs.

Earlier we argued that the borrowing and lending rates implicit in option prices will tend to be those applicable to professional market participants. Much the same is true for tax rates. Although these professionals do have the opportunity to place some securities in special accounts taxed at capital gains rates, most of their gains and losses are taxed as ordinary business income regardless of the source. When they consider a portfolio of stock and bonds that is equivalent to an option, they are looking at a portfolio that will provide the same *after-tax* returns as an option when both the call and all of the components of the portfolio are taxed at the rates for ordinary business income. The trading activities of these professionals will tend to make the market price of the option fairly close to the current cost of *their* equivalent portfolio. This is not necessarily to the detriment of individual investors; in fact, it may work in their favor. As an individual investor, you will consider *your* equivalent portfolio—the portfolio of stock and bonds that will give the same after-tax returns as an option when all of the securities involved are taxed at the rates applicable to you. Will the resulting portfolio always be different from that of the market professional? Not necessarily. It turns out that if your income from all sources is somehow taxed at a constant multiple of the rate applying to market professionals, then your portfolio will be the same. This would be the case, for example, for tax-exempt investors. But if some of your sources of income from securities transactions are taxed at a rate different from others, your equivalent portfolio will be different, as we will show in Chapter 6. You can then compare its cost with the market price of the option and take your position in the more favorable alternative. For example, suppose that you found that the market price of calls was consistently above that of your equivalent portfolio. Then you may be able to make consistently better after-tax returns with a strategy including some covered writing than you could using stock and bonds alone. At the same time, someone in a completely different tax situation might be able to achieve better after-tax returns by selling some calls. You would both gain at the expense of the government.[16]

So far we have looked at options and stocks and bonds as alternative investment vehicles and found that one or the other may be preferable for tax purposes. Options may also have tax advantages that are completely unrelated to any position we might want to take in the underlying stock. These advantages have survived a number of tax changes. Typically, it will be to our advantage to have the tax consequences of gains (not the gains themselves) postponed as long as possible and the tax consequences of losses taken as soon as possible. For example, suppose we have a current gain from, say, a real estate transaction. If we could delay the tax on this until next year or later, we will have the use of the tax money in the meantime. One way to do this would be to generate a comparable loss in the current tax year followed by an offsetting gain in the next tax year. Option spreads may provide a completely legal way of accomplishing this. Basically, you would want to take a long position in some option and a short position in another option on the same stock but with a different striking price or expiration date. By carefully choosing the proportions, you can keep the position as close to perfectly hedged as is permissible for tax purposes. You can then close out the unprofitable side of the spread just before the end of the current tax year and close out the profitable side just after the beginning of the next tax year. The main difficulty is that you cannot be sure how much the stock will move. If it stays put, the unprofit-

[16] If the Tax Reform Bill of 1984 becomes law, stock and stock options traded by options professionals will be treated as capital assets and any gain or loss will be taxed as if it were 60% long-term and 40% short-term. However, it appears that some of the transactions of options professionals who are also classified as dealers in stock may continue to be treated as generating ordinary income or loss. Our main conclusions would still hold: For stock options, professional traders will be taxed differently from individuals and will have different equivalent portfolios, thus providing the possibility that options will offer tax advantages.

able side may not generate much of a loss. But you can compensate for this by picking a volatile stock and taking a large position.[17]

5. OPTIONS MAY ALLOW LOWER TRANSACTIONS COSTS THAN THE STOCK. As was the case with borrowing and lending rates and with taxes, there are good reasons for thinking that it is the transactions costs faced by market professionals rather than by individual investors that are relevant in determining the market price of options. These costs are fairly low, so option prices may be very close to those that would prevail if there were no transactions costs at all.

At the same time, individual investors may face substantial transactions costs. These are described in Appendix 3B. Proper consideration of these costs will be essential in any investment decision. Once again, options markets may offer favorable opportunities. An investor would wish to compare the costs of obtaining a particular kind of portfolio for a particular length of time for the two alternatives: stock and bonds or options (and bonds, if necessary). All costs must be considered: those of setting up the position, maintaining it, and liquidating it; the explicit brokerage fee and the costs implicit in the bid-ask spread.

Typically we would find the following: If the position is to be held for a short period of time and will require frequent switching between stock and bonds if taken in that way, it will be cheaper to use options and bonds. On the other hand, if the position is to be held for a long time and would require only infrequent adjustment between stock and bonds, it will be cheaper to use the stock and bonds. This may, in part, explain the comparative popularity of shorter-term over the longest-term listed options.

6. OPTIONS MAY PROVIDE AN OPPORTUNITY TO USE CERTAIN KINDS OF SPECIAL KNOWLEDGE TO OBTAIN A PORTFOLIO WITH SUPERIOR PERFORMANCE—ONE THAT OFFERS A HIGHER EXPECTED RETURN THAN OTHER PORTFOLIOS WITH THE SAME DEGREE OF RISK. If an option is fairly priced, it will offer you an expected return appropriate for its degree of risk. If you are able to identify options that are undervalued or overvalued relative to the underlying stock, then you will have found superior investment opportunities—ones that offer a higher expected return than is justified by their risk. By combining these opportunities with fairly priced ones you can obtain an overall portfolio that has the amount of total risk you wish to bear while still providing superior performance. This is true even if you have no ability to identify undervalued or overvalued stocks. Of course, if you are able to do this as well, then you could obtain superior performance without using options.

[17] If the proposed law mentioned in footnote 16 is enacted, the opportunities for deferring taxes with spreads and hedges will be limited to certain covered writing positions. These positions would entail substantially more risk than the positions allowed under previous law.

But you could do even better by combining the two. For example, you could pick undervalued calls on undervalued stocks.

The essential requirement is the ability to pick options that are undervalued or overvalued relative to the stock. To do this, you need to have special information. You need to know something that is not already widely known and reflected in current market prices. One way to achieve this would be to have a special insight into how the fundamental variables we discussed fit together to determine exact option values. Alternatively, you might value options by widely known techniques, but have special knowledge about some of the determining factors. Certainly the striking price, the time to expiration, and the current stock price are available to everyone, but the other variables may offer better opportunities. The most promising of these is undoubtedly the volatility variable. What you want is an accurate prediction of the volatility of the underlying stock during the life of the option. Any special information about changes in the firm's investment or financing policies could lead to a better prediction than that being made by the market. For example, suppose that you are confident that a paper company will soon unexpectedly change its plans to sell the mineral rights on part of its land for a fixed fee and will instead take a large participatory interest in their development. You have no ideas about the likely success of this venture, but you do know that as a result the stock will be much more volatile in the future than the market had anticipated. You know that as soon as this becomes known, options will rise in price, relative to the stock. But you do not know how the stock price will respond. You could not take advantage of your information in the stock market, but you could in the options market. You could obtain superior performance, as we have defined it, by buying options. However, if you simply bought calls, you might lose money if the stock price fell, even though your analysis was correct. Similarly, if you only bought puts you might lose money if the stock price increased. So a better plan might be to buy both puts and calls with the same striking price. Or you could buy calls and sell some stock, or buy puts and buy some stock. Indeed, as suggested above, you might be able to adjust your portfolio to keep a neutral position in the stock, while still getting full benefit of your insights about volatility.

Another possibility is that you may be able to use publicly available information in a unique way to produce volatility forecasts that are better than those of the market. You may have a more efficient statistical method of extracting information from a series of past stock prices, or you may have a superior understanding of the relationship between future volatility and published accounting data. Such insights would be more valuable than information about a specific company, since they could potentially be used on all listed stocks simultaneously. This same advantage would accrue, to a lesser extent, to a special ability to forecast the volatility of the market. This would be useful information, since we would expect the volatilities of listed stocks as a group to move in the same way. It would be less useful than information about individual stocks, however, since we would need to form a diversified portfolio of options to take advantage of it. The volatility of any given stock might decrease, even though the volatility of the market as a whole increased.

Options may also offer the best opportunity to benefit from superior predictive ability about a firm's dividend policy. For example, suppose your analysis indicates that a firm will soon declare a completely unexpected sizeable cash dividend. It is not clear whether the market will interpret this as good news or bad news, so there is no sure way to make a profit with the stock. But you do know that when the announcement is made, the price of puts will rise relative to the underlying stock price, and the price of calls will fall. If you simply bought puts and wrote calls, you could lose money if the stock price rose in response to the news. But you could offset this by simultaneously taking an appropriate long position in the stock. The combined position would be hedged against stock price movements but would get full benefit from your information about dividends.

Since an unanticipated rise in interest rates will in general cause call prices to increase and put prices to decrease, it seems that you could also use options markets to profit from predictive ability about interest rates. This is true, but it would almost certainly be better to use this information directly in the bond markets or the financial futures markets.

Finally, you may have some special information about option values that does not require knowledge of valuation formulas or the inputs into such formulas. Various forms of technical analysis based on past price movements would be an example. For instance, your analysis might indicate that options are properly priced on average but tend to be overvalued after a large rise in the stock market and undervalued after a large fall. Naturally, such information could be used profitably in the options markets but not in the stock market.

In conclusion, it may be useful to recall that not all special information about the stock will favor the use of options. In deciding that a stock is undervalued or overvalued, you may feel that you have special information about its expected rate of return, or its volatility, or both. As we have just discussed, the latter may give you the ability to pick undervalued or overvalued options as well. But since the stock's expected rate of return may not be a separate determinant of option value, special information about it may be of no help whatsoever in spotting mispriced options. Of course, this information would influence the size of the total position you would want to take in the stock, but it would not in itself provide any reason for preferring any one way of taking that position to another.

7. Options may provide a means of hedging against unanticipated changes in stock volatility. Imagine the following situation. You feel that a particular stock is an excellent buy, so you have taken a substantial position in the stock. Nevertheless, you realize that its future price is uncertain, and that its volatility may unexpectedly increase. If this happens, your position will have more risk than you can afford to bear, so you will have to cut back and forego much of your potential profit. For protection, you might very well like to buy insurance against such an unexpected increase in volatility.

Options markets give you a way to do this. In this case, you might want to take the position in calls rather than directly in the stock. An unexpected increase in volatility will increase the value of the calls, and this will at least partly offset the foregone profits. More generally, by carefully selecting long positions in some options and short positions in others, you may be able to find a portfolio of options whose total value will be very sensitive to changes in volatility but relatively immune to changes in the stock price and other uncertainties. Such a portfolio would thus offer a pure opportunity to buy or sell insurance on volatility changes.

Of course, this strategy is very similar to the one we discussed earlier for taking advantage of special information about future volatility, but the motivation is different. A desire to hedge against certain kinds of risk does not imply, nor is it implied by, possession of special information.

8. Options may provide a way to hedge against unanticipated changes in a firm's dividend policy and a way to divide a stock's total return into separate dividend and price change components. Unexpected changes in a firm's dividend policy inevitably impose some costs on investors. For most people, there are only the small costs of minor portfolio adjustments to regain their preferred mix of capital gains income and dividend income. For others, the costs may be more severe. These individuals may well be interested in hedging against unexpected changes in dividend policy. Options provide a way to do this. For example, suppose an individual is the beneficiary of a trust fund that gives him the dividend income from large holdings in a few stocks. As is often the case, the stocks cannot be sold and the residual ownership will pass to another beneficiary. For this individual, a firm's decision to decrease dividends in favor of more price appreciation would be a major disaster, one he would like to insure himself against. To do this, he would like to find a portfolio whose total value is sensitive to dividend changes but is immune to other sources of risk. He could do this by taking a long position in calls combined with a properly chosen short position in stock and perhaps a few other calls. An unanticipated decrease in dividends will increase the value of

the purchased calls and provide the individual some compensation for his lost income. The remaining securities will make the value of the portfolio relatively insensitive to other sources of risk, including any changes in the stock price caused by the dividend announcement.

It might be argued that anyone with a large stake in dividends would be better advised to reduce his risk by selling off part of his position. This may be very difficult, but in principle options can give a way to do this as well. They can provide a way to separate the two parts of total return—dividends and price changes—into individual marketable components. Consider a covered writer of a European call with a zero striking price. He in fact owns only the dividends to be paid during the life of the option. In turn, the buyer of the call owns the entire price change component, but has no claim on the dividends. This latter division might be particularly attractive for an individual in a high tax bracket who would like to take a position in a stock paying large dividends. Another portfolio could also accomplish this same division. Suppose an individual buys a European call, sells a European put with the same striking price and expiration date, and makes a loan that will pay the amount of the striking price on the expiration date. A reexamination of the put-call parity relationship shows that he has purchased the entire price change component but owns no part of the dividend component. Of course, the difficulty is that European options are required to completely separate the two components. Because of the possibility of early exercise, American options will not do. Since European options are not currently traded on any exchange, the transaction would require a special arrangement in the over-the-counter market.

9. OPTIONS OFFER THE OPPORTUNITY TO AVOID CERTAIN IMPEDIMENTS TO THE SHORT SALE OF STOCK. We have already discussed ways in which options may provide advantages over a direct short sale: They may offer more favorable margin requirements and they may allow gains to be taxed at the long-term capital gains rate. There are two further advantages. Under current regulations, a stock can be sold short only after an up-tick in its price or after one or more zero-ticks preceded by an up-tick. In other words, the trade can take place only at a price higher than that of the last trade at a different price. In a declining market, some time may pass before an order can be filled. No such rule applies in the options market. An order to buy a put or sell a call can be filled immediately. Finally, for a stock to be sold short, it must be borrowed from its owner or a brokerage firm holding it in street name. The lender has the right to recall the stock at any time unless specific arrangements to the contrary have been made. In certain situations, it may be difficult to find the borrowed stock necessary to open the short position or maintain it after a recall. No such problems occur in the options market.

We conclude that options have much to offer. The success of the options exchanges is no accident; indeed, the puzzling thing is why it did not happen earlier. Similarly, the popularity of options on stock market indexes is exactly what economic arguments would have predicted. In fact, the reasons we gave suggest that there would be demand for an even broader menu of options than is currently available. Options with more frequent expiration dates and with longer maturities would be particularly useful. So would European options, in spite of lingering but questionable worries about their vulnerability to stock manipulation.

Also, individuals and firms may benefit from the existence of an options market even though they are not active participants. Publicly available price quotations on options may provide information that will be useful in other activities, and options may have other indirect beneficial effects on the allocation of resources. We will discuss this more fully in Chapter 8.

Finally, we must note that we have not yet mentioned a potential source of competition for options—convertible securities. These securities, such as convertible bonds, convertible preferred stock, and warrants, have many optionlike features. Indeed, we will show in Chapter 7 that nearly all corporate securities can be considered as packages of options on the assets of the firm. Consequently, convertible securities may offer many of the same advantages as options. But they are sufficiently different that the two are really complements rather than competitors. Although our discussion of convertible securities in Chapter 7 will be brief, we hope the analogies with options will show the possibilities they may provide.

APPENDIX 2A
The Relationship Between Options and Forward and Futures Contracts

Forward contracts are often confused with options. A *forward contract* is an arrangement whereby the seller currently agrees to deliver to the buyer a specified asset on a specified future date at a fixed price, to be paid on the *delivery date.* A long position in a forward contract and a European call are thus somewhat similar: both involve exchanging the underlying asset for a specified amount of money on a specified future date. However, there is a critical difference. The owner of a forward contract is committed to make this exchange; the owner of a call has the right, but not the obligation, to do so.

If the fixed price to be paid on the delivery date were sufficiently low, the buyer would have to pay a positive amount for the contract. If it were set high enough, the seller would have to pay the buyer to take the contract. Clearly, there is an intermediate price, known as the *forward price*, at which the current value of the contract would be zero. This is the fixed price that is customarily used for newly-written forward contracts. Consequently, a forward contract will have a value of zero when the contract is initiated. Of course, the value of an outstanding contract will subsequently change as the value of the underlying asset changes. On the delivery date, the value of the contract will be $S^* - F$, where S^* is the value of the underlying asset on the delivery date and F is the forward price at the time the contract was initiated. In contrast, the value of a call expiring on the same date would be $\max[0, S^* - K]$, where K is the striking price.

Although options and forward contracts have quite different payoffs, there is an interesting connection between them. To focus on the main issues, we will ignore the effects of transactions costs, margin requirements, and taxes. Table 2A-1 shows that under these conditions, *a newly-written forward contract is equivalent to a portfolio consisting of one purchased European call option on the underlying asset and one written European put option on the underlying asset, both with a common expiration date equal to the delivery date, and both with a common striking price equal to the forward price.*

Consequently, if there are to be no arbitrage opportunities, *the forward price must be the striking price that equates the value of the put and the call.* If C were less than P, we could lock in a sure profit by buying the call, selling the put, and selling a forward contract. This position would give an immediate cash inflow of $P - C$ and would never require any subsequent cash outflows. If C were greater than P, we could lock in a sure profit by selling the call, buying the put, and buying a forward contract.

Note that none of these results requires any information about the characteristics of the underlying asset. The conclusions are equally valid for

Table 2A-1
THE RELATIONSHIP BETWEEN OPTIONS AND FORWARD CONTRACTS

		Delivery Date and Expiration Date	
	Current Date	$S^* \leq F$	$S^* > F$
Buy forward contract	0	$S^* - F$	$S^* - F$
Buy call with $K = F$	$-C$	—	$S^* - F$
Sell put with $K = F$	P	$S^* - F$	—
Total	$P - C$	$S^* - F$	$S^* - F$

a common stock whose owner receives dividends and a commodity whose owner must pay storage costs. In fact, the same arguments would apply even if the underlying asset does not currently exist, as might be the case with a perishable commodity before the next harvest.

If the underlying asset is a common stock that does not pay dividends, then we can use the put-call parity relationship for European options to determine the forward price F. From Section 2-2, we know that

$$C - P = S - Kr^{-t}.$$

From our arbitrage analysis, we know that the forward price F is that value of the striking price K for which the put and call have the same value. Hence, $C = P$ and

$$F = K = Sr^{t}.$$

Now consider the case where the stock is paying cash dividends. In Section 2-2, we found that the put-call parity relationship would then be

$$C - P = S - D - Kr^{-t},$$

where D is the sum of the present values of all anticipated cash dividends with ex-dividend dates prior to the expiration date of the options. Our previous analysis then implies

$$F = (S - D)r^{t}.$$

In Section 2-2, we argued in the following way: If we know the market value of the dividends, then in the absence of arbitrage opportunities we can find the relative market prices of the options. Of course, we could have reversed this argument: If we know the market prices of the options, we can find the market price of the dividends,

$$D = P - C + S - Kr^{-t}.$$

As mentioned in our eighth reason for using options, we could purchase the right to receive all dividends paid during the life of the options by buying the stock, buying the put, selling the call, and borrowing Kr^{-t}. By applying our previous conclusions, we can infer the same information and accomplish the same result with a forward contract. The market value of the dividends can be written as

$$D = S - Fr^{-t}$$

and the ownership of the dividends can be obtained by buying the stock, selling a forward contract, and borrowing Fr^{-t}.

A futures contract is similar to a forward contract in many ways, but there is an important difference. An individual who takes a long position in a futures contract nominally agrees to buy a designated good or asset on the delivery date for the futures price prevailing at the time the contract is initiated. Hence, the futures price must also equal the spot price on the delivery date. Again, no money changes hands initially. Subsequently, however, as the futures price changes, the party in whose favor the price change occurred must immediately be paid the full amount of the change by the losing party. As a result, the payment required on the delivery date to buy the underlying good or asset is simply its spot price at that time. The difference between that amount and the initial futures price has been paid (or received) in installments throughout the life of the contract. Like the forward price, the equilibrium futures price must also continually change over time. It must do so in such a way that the remaining stream of future payments described above always has a value of zero.

In general, the continual resettlement feature of futures contracts makes it a difficult matter to determine an equilibrium futures price in terms of its underlying variables. However, if interest rates are nonstochastic and there are no arbitrage opportunities, it can be shown that futures prices are equal to forward prices. Consequently, the valuation formulas given for forward prices will then also hold for futures prices.

To see this, consider simultaneous forward and futures contracts on the same underlying asset with the same delivery date. Suppose that interest rates are nonstochastic and that the forward price is greater than the futures price. Then it would be possible to make an arbitrage profit with the following strategy. On the initial date, take a short position in a number of forward contracts equal to the total return that will be received from holding until its maturity a zero-coupon bond having the same maturity date as the contracts. With nonstochastic interest rates, this total return is the same as that which would be received from continually reinvesting in one-period bonds from the initial date until the delivery date. On each trading date, take a long position in a number of futures contracts equal to the total return received from continually reinvesting in one-period bonds from the initial date to the following trading date. Liquidate each of these futures positions on the following trading date and continually reinvest the (possibly negative) proceeds in one-period bonds until the delivery date.

By adding up the returns and remembering that the forward and futures prices must be equal on the delivery date, we would find that this strategy would produce an arbitrage profit proportional to the difference in the forward and futures prices. If the futures price were greater than the forward price, then an arbitrage profit could be obtained by reversing this strategy, so we can conclude that the two prices must be equal.

an exact option pricing formula

5

In the previous chapter, we developed some general propositions about option values. We showed that to prevent profitable riskless arbitrage opportunities, the value of an option must have certain relationships to the following variables:

1. Current stock price
2. Striking price
3. Time to expiration
4. Stock volatility
5. Interest rates
6. Cash dividends

These relationships took the form of inequalities and directional effects of each variable on option value. Only on the expiration date were we able to provide an equality relationship,

$$C = \max[0, S - K] \qquad \text{or} \qquad P = \max[0, K - S].$$

Prior to expiration, we could only say that the option value must lie within certain boundaries; we were not able to specify an exact formula between C or P and its determining variables.

The purpose of this chapter is to derive and analyze such an exact formula. To get such a precise result, we will now need more information than we did before. The required information characterizes the probability distribution of future stock prices and interest rates. As we will see, obtaining this information is not as formidable a task as it may first seem. It does not require superior forecasting ability, in the sense of being able to beat the market, nor does it require an understanding of the fundamental variables which cause stock prices to change.

Option pricing theory has a long and illustrious history, but it underwent a revolutionary change in 1973. At that time, Fischer Black and Myron Scholes presented the first completely satisfactory equilibrium option pricing model.[1] In the same year, Robert Merton, Professor of Finance at the Massachusetts Institute of Technology, extended their model in several important ways.[2] These path-breaking articles have formed the basis for many subsequent academic studies.

The mathematical tools employed in the Black-Scholes and Merton articles are quite advanced and have tended to obscure the underlying economic principles. Fortunately, William Sharpe, Professor of Finance at Stanford University, discovered a way to derive the same results using only elementary mathematics. His brilliant insight has the additional advantage of clearly showing the basic idea behind the model. In this chapter, we build on Sharpe's method and develop it into a complete model of option pricing.

Although each step in the argument can be easily understood, the length of the derivation may discourage many readers. To provide some motivation, and to illustrate the basic idea, we will first work through a simple numerical example.[3]

5-1. THE BASIC IDEA

Suppose the current price of a stock is $S = \$50$, and at the end of a period of time, its price must be either $S^* = \$25$ or $S^* = \$100$. A call on the stock is available with a striking price of $K = \$50$, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest. The

[1] Their celebrated article, "The Pricing of Options and Corporate Liabilities," appeared in the May-June 1973 issue of the *Journal of Political Economy*, pp. 637–659. Fischer Black is now Professor of Finance at the Massachusetts Institute of Technology and Myron Scholes is Professor of Finance at Stanford University.

[2] These results and many others are contained in Robert C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4 (Spring 1973), 141–183. Some additional results that are particularly relevant for our approach can be found in Robert C. Merton, "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem," *Journal of Financial Economics*, 5 (November 1977), 241–250.

[3] This chapter draws on an article by John Cox, Stephen Ross, and Mark Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7 (September 1979), 229–263.

one piece of information left unfurnished is the current value of the call, C. However, if profitable riskless arbitrage is not possible, we can deduce from the given information *alone* what the value of the call *must* be!

Consider forming the following levered hedge:

1. Write three calls at C each
2. Buy two shares at \$50 each
3. Borrow \$40 at 25%, to be paid back at the end of the period

Table 5-1 gives the return from this hedge for each possible level of the stock price at expiration. Regardless of the outcome, the hedge exactly breaks even on the expiration date. Therefore, to prevent profitable riskless arbitrage, the current cash flow from establishing the position must be zero; that is,

$$3C - 100 + 40 = 0.$$

Table 5-1
ARBITRAGE TABLE ILLUSTRATING THE FORMATION OF A RISKLESS HEDGE

	Current Date	*Expiration Date*	
		$S^* = 25$	$S^* = 100$
Write 3 calls	$3C$	—	−150
Buy 2 shares	−100	50	200
Borrow	40	−50	−50
Total		—	—

The current value of the call must then be $C = \$20$.

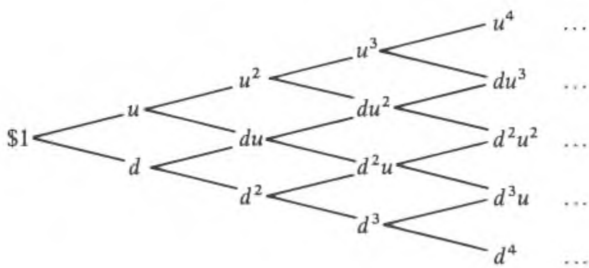
If the call were not priced at \$20, a sure profit would be possible. In particular, if $C = \$25$, the hedge in Table 5-1 would yield a current amount of \$15 and would experience no further gain or loss in the future. On the other hand, if $C = \$15$, then the same thing could be accomplished by buying three calls, selling short two shares, and lending \$40.

Table 5-1 can be interpreted as demonstrating that *an appropriately levered position in stock will replicate the future returns of a call.* That is, if we buy shares and borrow against them in the right proportion, we can, in effect, duplicate a pure position in calls. In view of this, it should seem less surprising that all we needed to determine the *exact* value of the call was its *striking price, underlying stock price, range of movement in the underlying stock price, and the rate of interest.* What may seem more incredible is what we do not need to know: Among other things, *we do not need to know the probability that the stock price will rise or fall.* Bulls and bears must agree on the value of the call, relative to its underlying stock price.[4]

This example is very simple, but it shows several essential features of option pricing. And we will soon see that it is not as unrealistic as it seems.

5-2. BINOMIAL RANDOM WALKS

Before we can derive an exact formula, we will need to develop some elementary statistical concepts. Suppose you play a game of chance in which, on n successive turns, you draw a single ball from an opaque urn containing 100 balls, of which k are black and $100 - k$ are red. After each drawing, you replace the ball drawn, so you always draw from an urn of the same composition. According to the rules, you can bet only at the beginning of the game. Thereafter, for every \$1 initially bet, you receive on each turn \$$u$ for every dollar accumulated up to then if you draw a black ball and \$$d$ if you draw a red ball, where $u > d$. To try it out, you decide to bet \$1.00. For this case, the possible outcomes after each of the first four drawings are represented in the following tree diagram:



For example, if $u = 1.1$ and $d = .9$, if you were fortunate to draw four black balls in a row, your bet would have grown to $u^4 = (1.1)^4 \approx \$1.46$. On the other hand, had you drawn a black followed by two reds and a black, you would have accumulated $uddu = u^2d^2 = (1.1)^2(.9)^2 \approx \$0.98$, netting you a 2¢ loss.

A first step in analyzing this game is counting, for a given total number of drawings n, the number of each possible outcome. If $n = 0$ (that is, you decide not to play and keep your dollar), you have one outcome of \$1; if $n = 1$, you have one d outcome and one u outcome; if $n = 2$, you have one d^2, two ud, and one u^2; and so on. A convenient way to represent these

[4] This provides an example of our earlier observation that the expected rate of growth of the stock price may not be a direct determinant of option value.

results is in the following array, known as Pascal's Triangle:

$n \backslash j$	0	1	2	3	4	5	6	...
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	...
⋮			⋮					

For example, row $n = 4$ represents one d^4, four ud^3, six u^2d^2, four u^3d, and one u^4. From the tree diagram, it is easily confirmed that this is the correct enumeration. Observe that an interior number in any row can be generated by summing the two numbers above and to the left in the row immediately above it. The sum of the numbers in each row is 2^n. Most important, there is a formula for representing any element in the array in terms of its row and column numbers (n, j):

$$\frac{n!}{j!(n-j)!},$$

where $n! \equiv n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$.[5] The numbers in the triangle are called binomial coefficients because they appear in the algebraic expansion of $(a + b)^n$.

Let us represent the outcome after n drawings as X_n. For example, if $n = 3$, then $X_3 = d^3, ud^2, u^2d$, or u^3. More generally,

$$X_n = d^n, ud^{n-1}, u^2d^{n-2}, \ldots, u^{n-2}d^2, u^{n-1}d, \quad \text{or} \quad u^n,$$

or, equivalently, $X_n = u^j d^{n-j}$ for $j = 0, 1, 2, \ldots, n$. Since we do not know in advance what value X_n will have for each drawing $n > 0$, we call X_n a *stochastic process.*

Since the urn contains exactly k black balls and $100 - k$ red balls at every drawing, the chance of drawing a black ball is $q \equiv k/100$, which we call the *probability* of a black ball. $1 - q \equiv (100 - k)/100$ is the probability of drawing a red ball. The probability q satisfies $0 \leq q \leq 1$ and the sum of the probabilities of each possibility, u and d, equals 1.

[5] $0! \equiv 1$.

In general, the probability q at any drawing n can depend on at least two things:

1. The number of previous drawings: $n - 1$
2. The sequence of previous outcomes: $X_0, X_1, \ldots, X_{n-1}$

For example, (1) would be true if we knew at the start of the game that $q = \frac{1}{2}$ at drawing $n = 10$ and $q = \frac{1}{4}$ at drawing $n = 13$; (1) would be false if $q = \frac{1}{3}$ for every drawing n; (2) would be true if, after a sequence d, d, and d in the first three drawings, the probability of u in the fourth were greater than if it were instead preceded by the sequence u, u, and u. For our game, q does not depend on (1) or (2). In other words, for every drawing, the urn has the same composition. For that reason, the stochastic process X_n is said to follow a *stationary multiplicative random walk.* Equivalently, we say the successive outcomes (X_1/X_0), (X_2/X_1), $\ldots$, (X_n/X_{n-1}) are independently and identically distributed.

Since successive drawings have this property, the outcomes for $n = 2$ of d^2, du, ud, and u^2 have associated accumulated probabilities of $(1 - q)^2$, $(1 - q)q$, $q(1 - q)$, and q^2. Observe that whatever the value of q, each accumulated probability is between 0 and 1, and their sum $(1 - q)^2 + 2q(1 - q) + q^2 = 1$. In general, the probability of any *one sequence* containing j drawings of black balls and $n - j$ red balls is $q^j(1 - q)^{n-j}$. Since there are exactly $n!/[j!(n - j)!]$ ways of this occurring, the probability of outcome X_n is

$$\left(\frac{n!}{j!(n-j)!}\right)q^j(1 - q)^{n-j}.$$

Moreover,[6]

$$\sum_{j=0}^{n}\left(\frac{n!}{j!(n-j)!}\right)q^j(1 - q)^{n-j} = 1.$$

For a given $j = a$, what is the probability that $X_n \geq u^a d^{n-a}$? Since $u^j d^{n-j}$ increases as j increases, we have

$$\Phi[a; n, q] \equiv \sum_{j=a}^{n}\left(\frac{n!}{j!(n-j)!}\right)q^j(1 - q)^{n-j},$$

the *complementary binomial distribution function.*

[6] $\sum$ is a shorthand notation for summation. For example,

$$\sum_{j=0}^{n} x_j \equiv x_0 + x_1 + x_2 + \cdots + x_n.$$

5-3. THE BINOMIAL OPTION PRICING FORMULA

In this section, we will develop the framework illustrated in the example into a complete valuation method. We begin by assuming that the stock price follows a multiplicative binomial process over discrete periods. The movement of the stock price will thus be essentially the same as the simple game described in the previous section. The rate of return on the stock over each period can have two possible values: $u - 1$, with probability q, or $d - 1$, with probability $1 - q$. For the moment, we assume the stock pays no dividends. If the current stock price is S, the stock price at the end of the period will thus be either uS or dS. We can represent this movement with the following diagram:

$$S \begin{cases} uS & \text{with probability } q \\ dS & \text{with probability } 1 - q. \end{cases}$$

We also assume that the interest rate is constant and positive. To focus on the basic issues, we will continue to assume that there are no taxes, transaction costs, or margin requirements. Hence, individuals are allowed to sell short any security and receive full use of the proceeds.[7] Furthermore, we assume that markets are competitive: A single individual can buy or sell as much of any security as he wishes without affecting its price.

Letting r denote one plus the interest rate over one period, we require $u > r > d$. If these inequalities did not hold, there would be profitable riskless arbitrage opportunities involving only the stock and riskless borrowing and lending.[8] For example, if $u > d > r$, an investor could make a certain profit on no investment by borrowing at r and buying the stock.

To see how to value a call on this stock, we start with the simplest situation: The expiration date is just one period away. Let C be the current value of the call, C_u be its value at the end of the period if the stock price goes to uS, and C_d be its value at the end of the period if the stock price goes to dS. Since there is now only one period remaining in the life of the call, we know that the terms of its contract and a rational exercise policy imply that $C_u = \max[0, uS - K]$ and $C_d = \max[0, dS - K]$. Therefore,

$$C \begin{cases} C_u = \max[0, uS - K] & \text{with probability } q \\ C_d = \max[0, dS - K] & \text{with probability } 1 - q. \end{cases}$$

[7] Of course, restitution is required for payouts made to securities held short.

[8] We will ignore the uninteresting special case where q is zero or one and $u = d = r$.

Suppose we form a portfolio containing Δ shares of stock and the dollar amount B in riskless bonds.[9] This will cost $S\Delta + B$. At the end of the period, the value of this portfolio will be

$$S\Delta + B \begin{cases} uS\Delta + rB & \text{with probability } q \\ dS\Delta + rB & \text{with probability } 1-q. \end{cases}$$

Since we can select Δ and B in any way we wish, suppose we choose them to equate the end-of-period values of the portfolio and the call for each possible outcome. This requires that

$$uS\Delta + rB = C_u,$$
$$dS\Delta + rB = C_d.$$

Solving these equations, we find

$$\Delta = \frac{C_u - C_d}{(u-d)S}, \qquad B = \frac{uC_d - dC_u}{(u-d)r}. \tag{1}$$

With Δ and B chosen in this way, we have what we referred to in Section 2-3 as an *equivalent portfolio.*

If there are to be no riskless arbitrage opportunities, the current value of the call, C, cannot be less than the current value of the equivalent portfolio, $S\Delta + B$. If it were, we could make a riskless profit with no net investment by buying the call and selling the portfolio. It is tempting to say that it also cannot be worth more, since then we would have a riskless arbitrage opportunity by reversing our procedure and selling the call and buying the portfolio. But this overlooks the fact that the person who bought the call we sold has the right to exercise it immediately.

Suppose that $S\Delta + B < S - K$. If we try to make an arbitrage profit by selling calls for more than $S\Delta + B$, but less than $S - K$, then we will soon find that we are the source of arbitrage profits rather than their recipient. Anyone could make an arbitrage profit by buying our calls and exercising them immediately.

We might hope that we will be spared this embarrassment because everyone will somehow find it advantageous to hold the calls for one more period as an investment rather than take a quick profit by exercising them

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[9] Buying bonds is the same as lending; selling them is the same as borrowing.

immediately. But each person will reason in the following way. If I do not exercise now, I will receive the same payoff as a portfolio of Δ shares of stock and B in bonds. If I do exercise now, I can take the proceeds, $S - K$, buy this same portfolio and some extra bonds as well, and have a higher payoff in every possible circumstance. Consequently, no one would be willing to hold the calls for one more period.

Summing up all of this, we conclude that if there are to be no riskless arbitrage opportunities, it must be true that

$$\begin{aligned} C &= S\Delta + B \\ &= \frac{C_u - C_d}{u - d} + \frac{uC_d - dC_u}{(u - d)r} \\ &= \left[\left(\frac{r - d}{u - d}\right)C_u + \left(\frac{u - r}{u - d}\right)C_d\right]\Big/ r \end{aligned} \tag{2}$$

if this value is greater than $S - K$, and if not, $C = S - K$.[10]

Equation (2) can be simplified by defining $p \equiv (r - d)/(u - d)$, so that $1 - p = (u - r)/(u - d)$ and we can write

$$C = [pC_u + (1 - p)C_d]/r. \tag{3}$$

It is easy to see that in the present case, with no dividends, this will always be greater than $S - K$ as long as the interest rate is positive.[11] Hence, Equation (3) is the exact formula for the value of a call one period prior to expiration in terms of S, K, u, d, and r.

This formula has a number of notable features. First, the probability q does not appear in the formula. This means, surprisingly, that even if different investors have different subjective probabilities about an upward or downward movement in the stock, they could still agree on the relationship of C to S and r.

Second, the value of the call does not depend on investors' attitudes toward risk. In constructing the formula, the only assumption we made about an individual's behavior was that he prefers more wealth to less wealth and therefore has an incentive to take advantage of profitable riskless arbitrage opportunities. We would obtain the same formula whether investors are risk averse or risk preferring.

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[10] Our discussion could be easily modified to include European calls. Since immediate exercise is then precluded, their value would always be given by Equation (2), even if this is less than $S - K$.

[11] To confirm this, note that if $uS \le K$, then $S < K$ and $C = 0$, so $C > S - K$. Also if $dS \ge K$, then $C = S - (K/r) > S - K$. The remaining possibility is $uS > K > dS$. In this case, $C = p(uS - K)/r$. This is greater than $S - K$ if $(1 - p)dS < (r - p)K$, which is certainly true as long as $r > 1$.

Third, the only random variable on which the call value depends is the stock price itself. In particular, it does not depend on the random prices of other securities or portfolios, such as the market portfolio containing all securities in the economy. If another pricing formula involving other variables was submitted as giving equilibrium market prices, we could immediately show that it was incorrect by using our formula to make riskless arbitrage profits while trading at those prices.

It is easier to understand these features if it is remembered that the formula is only a relative pricing relationship giving C in terms of S, u, d, and r. Investors' attitudes toward risk and the characteristics of other assets may indeed influence call values indirectly, through their effect on these variables, but they will not be separate determinants of call value.

Finally, observe that $p \equiv (r - d)/(u - d)$ is always greater than zero and less than one, so it has the properties of a probability. In fact p is the value q would have in equilibrium if investors were risk neutral.[12] To see this, note that the expected rate of return on the stock, which is the sum of each possible rate of return times its probability of occurring, would then be the riskless interest rate, so

$$q(uS) + (1 - q)(dS) = rS$$

and $q = (r - d)/(u - d) = p$. Hence, the value of the call can be interpreted as the expectation of its discounted future value in a risk-neutral world.[13] In light of our earlier observations, this is not surprising. Since the formula does not involve q or any measure of attitudes toward risk, then it must be the same for any set of preferences, including risk neutrality.

It is important to note that this does not imply that the equilibrium expected rate of return on the call is the riskless interest rate. Indeed, our argument has shown that, in equilibrium, holding the call over the period is exactly equivalent to holding the equivalent portfolio. Consequently, the risk and expected rate of return of the call must be the same as that of the equivalent portfolio. As we will show in Section 5-5, $\Delta \geq 0$ and $B \leq 0$, so the equivalent portfolio is a particular levered long position in the stock. In equilibrium, the same is true for the call. Of course, if the call is currently mispriced, its risk and expected return over the period will differ from that of the equivalent portfolio.

A different interpretation of p and the valuation formula may also be helpful. In Chapter 8, we show that p/r is the value of a claim that will pay

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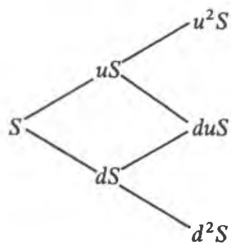
[12] We define a *risk-neutral investor* to be one who is indifferent between an investment with a certain rate of return and another investment with an uncertain rate of return which has the same expected value. He neither insists on being paid for bearing risk nor is he willing to pay others to let him bear risk.

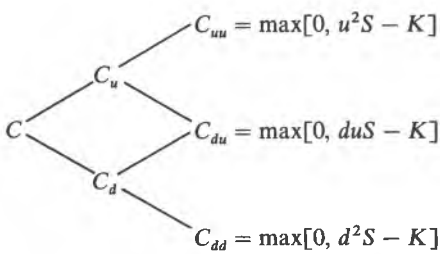
[13] This property was first noted by John Cox and Stephen Ross in "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3 (January–March 1976), 145–166.

one dollar at the end of the period if and only if the stock price moves to uS. Similarly, $(1 - p)/r$ is the value of a claim that will pay one dollar if and only if the stock price moves to dS. The payoff to a call is equivalent to that of a package containing C_u units of the first claim and C_d units of the second claim, so its value should be $[C_u(p/r)] + [C_d(1 - p)/r]$, which is exactly Equation (3).

Now we can consider the next simplest situation: a call with two periods remaining before its expiration date. In keeping with the binomial process, the stock can take on three possible values after two periods:



Similarly, for the call,



C_{uu} stands for the value of a call two periods from the current time if the stock price moves upward each period; C_{du} and C_{dd} have analogous definitions.

At the end of the current period there will be one period left in the life of the call and we will be faced with a problem identical to the one we just solved. Thus, from our previous analysis, we know that when there are two periods left,

$$C_u = [pC_{uu} + (1 - p)C_{ud}]/r \tag{4a}$$

and

$$C_d = [pC_{du} + (1 - p)C_{dd}]/r. \tag{4b}$$

Again we can select a portfolio of $S\Delta$ in stock and B in bonds whose end-of-period value will be C_u if the stock price goes to uS and C_d if the stock price goes to dS. Indeed, the functional form of Δ and B remains unchanged. To get the new values of Δ and B, we simply use Equation (1) with the new values of C_u and C_d.

Can we now say, as before, that an opportunity for profitable riskless arbitrage will be available if the current price of the call is not equal to the new value of this portfolio or $S - K$, whichever is greater? Yes, but there is an important difference. With one period to go, we could plan to lock in a riskless profit by selling an overpriced call and using part of the proceeds to buy the equivalent portfolio. At the end of the period, we knew that the market price of the call must have been equal to the value of the portfolio, so the entire position could have been safely liquidated at that point. But this was true only because the end of the period was the expiration date. Now we have no such guarantee. At the end of the current period, when there is still one period left, the market price of the call could still be in disequilibrium and be greater than the value of the equivalent portfolio. If we closed out the position then, selling the portfolio and repurchasing the call, we could suffer a loss that would more than offset our original profit. However, we could always avoid this loss by maintaining the portfolio for one more period. The value of the portfolio at the end of the current period will always be exactly sufficient to purchase the portfolio we would want to hold over the last period. In effect, we would have to readjust the proportions in the equivalent portfolio, but we would not have to put up any more money.

Consequently, we conclude that even with two periods to go, there is a strategy we could follow that would guarantee riskless profits with no net investment if the current market price of a call differs from the maximum of $S\Delta + B$ and $S - K$. Hence, the larger of these is the current value of the call.

Since Δ and B have the same functional form in each period, the current value of the call in terms of C_u and C_d will again be $C = [pC_u + (1 - p)C_d]/r$ if this is greater than $S - K$, and $C = S - K$ otherwise. By substituting from Equation (4) into the former expression, and noting that $C_{du} = C_{ud}$, we obtain

$$\begin{aligned} C &= [p^2 C_{uu} + 2p(1 - p)C_{ud} + (1 - p)^2 C_{dd}]/r^2 \\ &= \{p^2 \max[0, u^2 S - K] + 2p(1 - p) \max[0, duS - K] \\ &\quad + (1 - p)^2 \max[0, d^2 S - K]\}/r^2. \end{aligned} \tag{5}$$

A little algebra shows that this is always greater than $S - K$ if, as assumed, r is always greater than one, so this expression gives the exact value of the call.[14]

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[14] In the current situation, with no dividends, we know from Chapter 4 that the call should not be exercised before the expiration date. In the general case, with dividends, this is no longer true, and we must use the procedure of checking every period.

All of the observations made about Equation (3) also apply to Equation (5), except that the number of periods remaining until expiration, n, now emerges clearly as an additional determinant of the call value. For Equation (5), $n = 2$. That is, the full list of variables determining C is S, K, n, u, d, and r.

We now have a recursive procedure for finding the value of a call with any number of periods to go. By starting at the expiration date and working backwards, we can write down the general valuation formula for any n:

$$C = \left\{ \sum_{j=0}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} \max[0, u^j d^{n-j} S - K] \right\} \Big/ r^n. \tag{6}$$

This gives us the complete formula, but with a little additional effort we can express it in a more convenient way.

Let a stand for the minimum number of upward moves that the stock must make over the next n periods for the call to finish in-the-money. Thus a will be the smallest nonnegative integer such that $u^a d^{n-a} S > K$. By taking the natural logarithm of both sides of this inequality, we can write a as the smallest nonnegative integer greater than $\log(K/Sd^n)/\log(u/d)$.

For all $j < a$, $\max[0, u^j d^{n-j} S - K] = 0$ and for all $j \geq a$, $\max[0, u^j d^{n-j} S - K] = u^j d^{n-j} S - K$. Therefore,

$$C = \left\{ \sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} [u^j d^{n-j} S - K] \right\} \Big/ r^n.$$

Of course, if $a > n$, the call will finish out-of-the-money even if the stock moves upward every period, so its current value must be zero.

By breaking up C into two terms, we can write

$$\begin{aligned} C = S&\left[\sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} \left(\frac{u^j d^{n-j}}{r^n} \right) \right] \\ &- Kr^{-n} \left[\sum_{j=a}^{n} \left(\frac{n!}{j!(n-j)!} \right) p^j (1-p)^{n-j} \right]. \end{aligned}$$

Now, the latter bracketed expression is the complementary binomial distribution function $\Phi[a; n, p]$. The first bracketed expression can also be interpreted as a complementary binomial distribution function $\Phi[a; n, p']$, where

$$p' \equiv (u/r)p \qquad \text{and} \qquad 1 - p' = (d/r)(1-p).$$

p' is a probability, since $0 < p' < 1$. To see this, note that $p < (r/u)$ and

$$p^j(1-p)^{n-j}\left(\frac{u^j d^{n-j}}{r^n}\right) = [(u/r)p]^j[(d/r)(1-p)]^{n-j} = p'^j(1-p')^{n-j}.$$

We can summarize our development of the Sharpe binomial method up to this point in the following formula:

BINOMIAL OPTION PRICING FORMULA

$$C = S\Phi[a; n, p'] - Kr^{-n}\Phi[a; n, p]$$

where

$$p \equiv (r-d)/(u-d) \qquad \text{and} \qquad p' \equiv (u/r)p$$

$a \equiv$ the smallest nonnegative integer greater than

$$\log(K/Sd^n)/\log(u/d).$$

If $a > n$, $C = 0$.

It is now clear that all of the comments we made about the one period valuation formula are valid for any number of periods. In particular, the value of a call should be the expectation, in a risk-neutral world, of the discounted value of the payoff it will receive. In fact, that is exactly what Equation (6) says. Why, then, should we waste time with the recursive procedure when we can write down the answer in one direct step? The reason is that while this one-step approach is always technically correct, it is really useful only if we know in advance the circumstances in which a rational individual would prefer to exercise the call before the expiration date. If we do not know this, we have no way to compute the required expectation. In the present example, a call on a stock paying no dividends, it happens that we can determine this information from other sources: The call should never be exercised before the expiration date. As we will see in Section 5-9, with puts or with calls on stocks which pay dividends, we will not be so lucky. Finding the optimal exercise strategy will be an integral part of the valuation problem. The full recursive procedure will then be necessary.

5-4. RISKLESS TRADING STRATEGIES

The following numerical example illustrates how we could use the formula if the current *market price M* ever diverged from its *formula value C*. If

$M > C$, we would hedge, and if $M < C$, "reverse hedge," to try and lock in a profit. Suppose the values of the underlying variables are

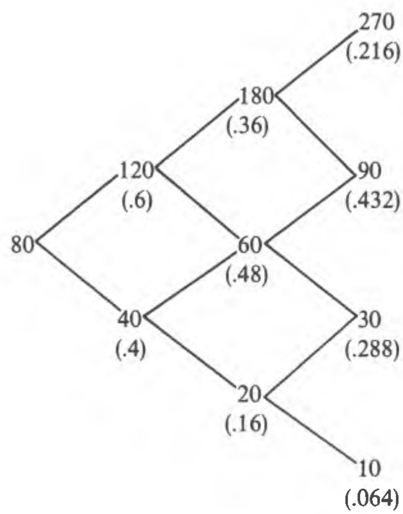
$$S = 80, \quad n = 3, \quad K = 80, \quad u = 1.5, \quad d = .5, \quad r = 1.1.$$

In this case, $p = (r - d)/(u - d) = .6$. The relevant values of the discount factor are

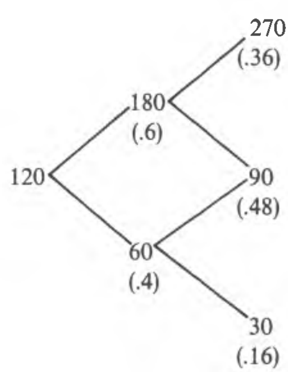
$$r^{-1} = .909, \qquad r^{-2} = .826, \qquad r^{-3} = .751.$$

The paths the stock price may follow and their corresponding probabilities (using probability p) are:

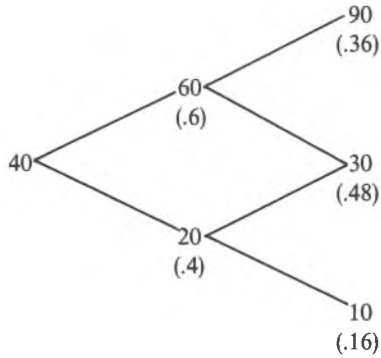
When $n = 3$, with $S = 80$,



When $n = 2$, if $S = 120$,



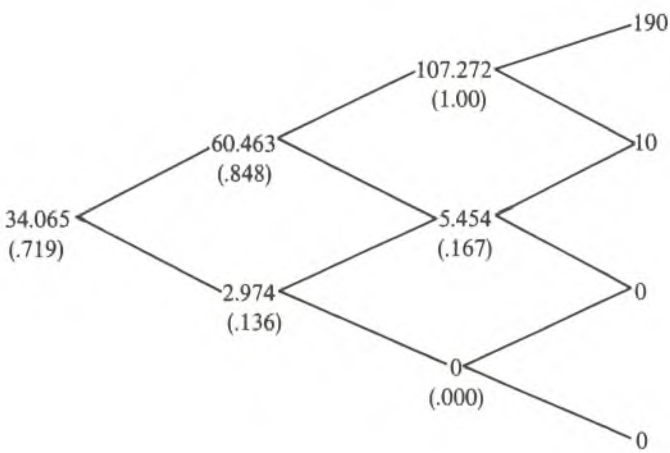
When $n = 2$, if $S = 40$,



Using the formula, the current value of the call would be

$$C = .751[.064(0) + .288(0) + .432(90 - 80) + .216(270 - 80)] = 34.065.$$

Recall that to form a riskless hedge, for each call we sell, we buy and subsequently keep adjusted a portfolio containing $S\Delta$ in stock and B in bonds, where $\Delta = (C_u - C_d)/(u - d)S$. The following tree diagram gives the paths the call value may follow and the corresponding values of Δ:



With this preliminary analysis, we are prepared to use the formula to take advantage of mispricing in the market. Suppose that when $n = 3$, the market price of the call is 36. Our formula tells us the call should be worth

34.065. The option is overpriced, so we could plan to sell it and assure ourselves of a profit equal to the mispricing differential. Here are the steps you could take for a typical path the stock might follow.

STEP 1 ($n = 3$): Sell the call for 36. Take 34.065 of this and invest it in a portfolio containing $\Delta = .719$ shares of stock by borrowing $.719(80) - 34.065 = 23.455$. Take the remainder, $36 - 34.065 = 1.935$, and put it in the bank.

STEP 2 ($n = 2$): Suppose the stock goes to 120 so that the new Δ is .848. Buy $.848 - .719 = .129$ more shares of stock at 120 per share for a total expenditure of 15.480. Borrow to pay the bill. With an interest rate of .1, you already owe $23.455(1.1) = 25.801$. Thus, your total current indebtedness is $25.801 + 15.480 = 41.281$.

STEP 3 ($n = 1$): Suppose the stock price now goes to 60. The new Δ is .167. Sell $.848 - .167 = .681$ shares at 60 per share, taking in $.681(60) = 40.860$. Use this to pay back part of your borrowing. Since you now owe $41.281(1.1) = 45.409$, the repayment will reduce this to $45.409 - 40.860 = 4.549$.

STEP 4*d* ($n = 0$): Suppose the stock price now goes to 30. The call you sold has expired worthless. You own .167 shares of stock selling at 30 per share, for a total value of $.167(30) = 5$. Sell the stock and repay the $4.549(1.1) = 5$ that you now owe on the borrowing. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3 = 2.575$.

STEP 4*u* ($n = 0$): Suppose, instead, the stock price goes to 90. The call you sold is in the money at the expiration date. Buy back the call, or buy one share of stock and let it be exercised, incurring a loss of $90 - 80 = 10$ either way. Borrow to cover this, bringing your current indebtedness to $5 + 10 = 15$. You own .167 shares of stock selling at 90 per share, for a total value of $.167(90) = 15$. Sell the stock and repay the borrowing. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^3 = 2.575$.

In summary, if we were correct in our original analysis about stock price movements (which did not involve the unenviable task of predicting whether the stock price would go up or down), and if we faithfully adjust our portfolio as prescribed by the formula, then we can be assured of walking away in the clear at the expiration date, while still keeping the original differential and the interest it has accumulated. It is true that closing out the position before the expiration date, which involves buying back the option at its then current market price, might produce a loss that would more than offset our profit, but this loss could always be avoided by waiting until the expiration date. Moreover, if the market price comes into line with the formula value before the expiration date, we can close out the position then with no loss and be rid of the concern of keeping the portfolio adjusted.

It still might seem that we are depending on rational behavior by the person who bought the call we sold. If instead he behaves foolishly and exercises at the wrong time, could he make things worse for us as well as for himself? Fortunately, the answer is no. Mistakes on his part can only mean greater profits for us. Suppose that he exercises too soon. In that circumstance, the equivalent portfolio will always be worth more than $S - K$, so we could close out the position then with an extra profit.

Suppose, instead, that he fails to exercise when it would be optimal to do so. Again there is no problem. Since exercise is now optimal, our equivalent portfolio will be worth $S - K$.[15] If he had exercised, this would be exactly sufficient to meet the obligation and close out the position. Since he did not, the call will be held at least one more period, so we calculate the new values of C_u and C_d and revise our equivalent portfolio accordingly. But now the amount required for the portfolio, $S\Delta + B$, is less than the amount we have available, $S - K$. We can withdraw these extra profits now and still maintain the equivalent portfolio. The longer the holder of the call goes on making mistakes, the better off we will be.

Consequently, we can be confident that things will eventually work out right no matter what the other party does. The return on our total position, when evaluated at prevailing market prices at intermediate times, may be negative. But over a period ending no later than the expiration date, it will be positive.

In conducting the hedging operation, the essential thing was to maintain the proper proportional relationship: For each call we are short, we hold Δ shares of stock and the dollar amount B in bonds in the equivalent portfolio. To emphasize this, we will refer to the number of shares held for each call as the *neutral position ratio.* In our example, we kept the number of calls constant and made adjustments by buying or selling stock and bonds. As a result, our profit was independent of the market price of the call between the time we initiated the hedge and the expiration date. If things got worse before they got better, it did not matter to us.

Instead, we could have made the adjustments by keeping the number of shares of stock constant and buying or selling calls and bonds. However, this could be dangerous. Suppose that after initiating the position, we

[15] If we were reverse hedging by buying an undervalued call and selling the equivalent portfolio, then we would ourselves want to exercise at this point. Since we will receive $S - K$ from exercising, this will be exactly enough money to buy back the equivalent portfolio.

needed to increase the neutral position ratio to maintain the proper proportions. This can be achieved in two ways: (1) buy more stock, or (2) buy back some of the calls. If we adjust through the stock, there is no problem. If we insist on adjusting through the calls, not only is the hedge no longer riskless, but it could even end up losing money! This can happen if the call has become even more overpriced. We would then be closing out part of our position in calls at a loss. To remain hedged, the number of calls we would need to buy back depends on their value, not their price. Therefore, since we are uncertain about their price, we then become uncertain about the return from the hedge. Worse yet, if the call price gets high enough, the loss on the closed portion of our position could throw the hedge operation into an overall loss.

To see how this could happen, let us rerun the hedging operation, where we adjust the hedge ratio by buying and selling calls.

STEP 1 ($n = 3$): Same as before.

STEP 2 ($n = 2$): Suppose the stock goes to 120, so that the new $\Delta = .848$. The call price has gotten further out of line and is now selling for 75. Since its value is 60.463, it is now overpriced by 14.537. With .719 shares, you must buy back $1 - .848 = .152$ calls to produce a hedge ratio of $.848 = .719/.848$. This costs $75(.152) = 11.40$. Borrow to pay the bill. With the interest rate of .1, you already owe $23.455(1.1) = 25.801$. Thus, your total current indebtedness is $25.801 + 11.40 = 37.201$.

STEP 3 ($n = 1$): Suppose the stock goes to 60 and the call is selling for 5.454. Since the call is now fairly valued, no further excess profits can be made by continuing to hold the position. Therefore, liquidate by selling your .719 shares for $.719(60) = 43.14$ and close out the call position by buying back .848 calls for $.848(5.454) = 4.625$. This nets $43.14 - 4.625 = 38.515$. Use this to pay back part of your borrowing. Since you now owe $37.20(1.1) = 40.921$, after repayment you owe 2.406. Go back to the bank and withdraw your original deposit, which has now grown to $1.935(1.1)^2 = 2.341$. Unfortunately, after using this to repay your remaining borrowing, you still owe .065.

Since we adjusted our position at Step 2 by buying overpriced calls, our profit is reduced. Indeed, since the calls were considerably overpriced, we actually lost money despite apparent profitability of the position at Step 1. We can draw the following adjustment rule from our experiment: *To adjust a hedged position, never buy an overpriced option or sell an underpriced option.* As a corollary, whenever we can adjust a hedged position by buying more of an underpriced option or selling more of an overpriced option, our profit will be enhanced if we do so. For example, at Step 3 in the original hedging illustration, had the call still been overpriced, it would have been better to adjust the position by selling more calls rather than selling stock. In summary, by choosing the right side of the position to adjust at intermediate dates, *at a minimum* we can be assured of earning the original differential and its accumulated interest, and we may earn considerably more.

Is it ever possible to work our way into a position where, to maintain neutrality, we are *forced* to buy an overpriced option or sell an underpriced option? This can never happen with a hedge, since, if necessary, we can always adjust with the stock. However, if we are careless, this can happen with spreads and combinations. To see this, suppose, at the initiation of a neutral purchased straddle, both the put and the call are underpriced. Each side of the straddle is then separately profitable. At a subsequent date, suppose both options remain underpriced and we need to increase the position ratio to maintain neutrality. We can do this without violating our rule by buying more calls. If, instead, the position ratio should be decreased, we can buy more puts. This position has no dangers.

However, suppose, at its initiation, the call is underpriced and the put overpriced. Despite this, the purchased straddle looks profitable since the calls are significantly underpriced, relative to the overpricing of the puts. At a subsequent date, suppose the call remains underpriced and the put overpriced. If we need to increase the position ratio, we can buy more of the underpriced calls—again, no problem. However, if we need to *decrease* the position ratio, we must either (1) sell back some calls, or (2) buy more puts. In either case, we are forced to violate our adjustment rule. Suppose that we did so by selling calls. If the calls are less underpriced than they were originally, then we will be giving up some of our potential profit, but we still will not be risking a loss. However, suppose the calls are more underpriced than they were originally. The potential loss from the sale could then indeed be greater than the original potential profit. However, market prices are now more out of line than ever, so it certainly seems that we could still insure a profit by adding another neutral straddle of large enough size to our original position. In effect, we would be increasing the scale of each side of our position. And we could keep doing this each period if necessary, knowing that market prices must come into line at the end of the last period. The problem is that the scale of our position may become so large that capital limitations or even the smallest mistake in analysis or implementation could lead to disaster.

We can avoid being pushed into this unfortunate position if we *never initiate a covered position where one side of the position is unfavorable*. For example, we should never put on a spread where both sides are overpriced or both are underpriced. We should not put on a combination where one side is overpriced and the other underpriced. As a corollary, *whenever one* *side of a covered position becomes unfavorable, we should liquidate that side and replace it with another option with a favorable price.* For instance, suppose we buy a straddle in underpriced calls and puts and, at a subsequent date, the puts become overpriced. We should immediately sell the puts and replace them with other underpriced purchased puts or other overpriced written calls. If neither are available, we can always short the stock.

To recapitulate, we have the following rules for initiating and maintaining neutral positions:

1. Never initiate a neutral position where one side of the position is unfavorable.
2. Whenever one side of a neutral position becomes unfavorable, liquidate that side and replace it with another option with a favorable price.
3. Never adjust by buying an overpriced option or selling an underpriced option.
4. If possible, always adjust by buying an underpriced option or selling an overpriced option.

Adhering faithfully to these rules ensures a profit of at least the original pricing differential and eliminates all ambiguity about which side of a position to adjust to maintain neutrality.

5-5. OPTION RISK AND EXPECTED RETURN

In this section, we show how the equilibrium risk and expected return of an option are related to the risk and expected return of the underlying stock. We will also derive the relationship of option pricing to the "capital asset pricing model," which is widely used in portfolio management. In particular, we show how to calculate the "alpha" and "beta" of an option.

It is important to note that this information will be derived from our previous results without additional assumptions (except in the calculation of alpha and beta). If we were interested only in valuing an option in terms of the stock, or in pursuing riskless arbitrage profits if market prices differed from this value, then we would not need the results of this section. However, if we plan to include options as part of an investment portfolio, then the risk and return analysis that follows is of critical importance.

Stock Risk and Expected Return. Over a single period, the total return on a security is its price at the end of the period, plus any cash distribution made at the end of the period, divided by its price at the beginning of the period. In our binomial model, the total return of the stock is either u or d.

Its *expected return*, m_S, is the weighted average of the possible total returns, where the weights are the respective probabilities. That is,

$$m_S \equiv qu + (1 - q)d.$$

One measure of stock risk is the variance of the total return, $v_S{}^2$. This is the weighted average of the squared deviations of the possible total returns from their mean, where the weights are the respective probabilities. That is,

$$v_S{}^2 \equiv q(u - m_S)^2 + (1 - q)(d - m_S)^2.$$

Substituting for the mean m_S from the previous equation, we can simplify this to

$$v_S = [q(1 - q)(u - d)^2]^{1/2},$$

where v_S, the square root of the variance, is called the *standard deviation*, or simply the stock *volatility*. Often these measures are expressed in terms of the rate of return, which is the total return minus one. It is easy to see that the expected rate of return is $m_S - 1$ and that the standard deviation of the rate of return is v_S.

Option Elasticity. Recall that the neutral hedge ratio is

$$\Delta = \frac{C_u - C_d}{(u - d)S}.$$

Suppose that we think of the stock price as having moved downward and then ask: What would be the change in the value of the call relative to the change in the value of the stock if the stock had instead moved upward? This is exactly what Δ tells us. If we wish to make this comparison in terms of percentage changes, then we would divide the numerator of Δ by the current call value C, and the denominator by the current stock price S. This concept is called the option's *elasticity* and will be denoted by Ω. That is,

$$\Omega \equiv (S/C)\Delta.$$

For a put, $\Delta = (P_u - P_d)/(u - d)S$ and $\Omega = (S/P)\Delta$. Since $P_u \leq P_d$, both Δ and Ω are less than or equal to zero. One further fact, which we will state but not prove, will be useful later: For both puts and calls, Ω increases as K increases.

Option Risk. We can apply these same measures to an option. The mean m_C and standard deviation v_C of the total return of a call over one period are computed in the same way as the corresponding statistics for the stock:

$$m_C \equiv \frac{qC_u + (1-q)C_d}{C} \qquad \text{and} \qquad v_C \equiv \left[q(1-q)\left(\frac{C_u - C_d}{C}\right)^2\right]^{1/2}.$$

By combining our equations for v_S and v_C and using the definitions of Δ and Ω, we find that

$$v_C = \Omega v_S.$$

This equation relates the risk of a call to the risk of the underlying stock. The *risk of a call* (the standard deviation of its rate of return) *equals its elasticity times its underlying stock volatility.* The elasticity Ω can be easily computed, since it requires knowing only u, d, C, C_u, and C_d.

Moreover, it is easy to show that in percentage terms (rates of return), *the call can never be less risky than the stock.* That is, $v_C \geq v_S$. To demonstrate this, we must show that $\Omega \geq 1$. In Section 5-3 we showed that

$$C = \frac{pC_u + (1-p)C_d}{r} \qquad \text{where} \quad p \equiv \frac{r-d}{u-d}.$$

This implies

$$r[C_u - C_d - (u-d)C] + [uC_d - dC_u] = 0.$$

If the second bracketed expression is nonpositive, then the first is nonnegative. By the definition of Ω,

$$C_u - C_d - (u-d)C \geq 0 \qquad \text{if and only if} \quad \Omega \geq 1.$$

Therefore, if we can show $uC_d - dC_u$ is nonpositive, then we have proved $\Omega \geq 1$.

From our earlier development in Section 5-3, we know that one period from the present the call value will be either

$$C_d = \{E \max[0, dSu^j d^{n-1-j} - K]\}/r^{n-1},$$

or

$$C_u = \{E \max[0, uSu^j d^{n-1-j} - K]\}/r^{n-1},$$

where E represents the expected value with respect to the probability distribution for j when $q = p$. After substituting these expressions, it should be clear that $uC_d - dC_u \leq 0$, so that we have confirmed our result.

Note that $\Omega \geq 1$ implies that $C - S\Delta \leq 0$. Since $B = C - S\Delta$, this verifies our earlier comment that $B \leq 0$ and hence our conclusion that over a single period the call is equivalent to a particular levered long position in the stock. This, of course, squares with our result about option risk and stock risk, since in rate of return terms a levered portfolio is more risky than an unlevered one.

The mean m_P and standard deviation v_P of the total return of a put over one period are defined in the same way as for a call:

$$m_P = \frac{qP_u + (1-q)P_d}{P} \qquad \text{and} \qquad v_P = \left[q(1-q)\frac{(P_u - P_d)^2}{P}\right]^{1/2}.$$

The volatility of a put can then be written as

$$v_P = -\Omega v_S.$$

The minus sign is necessary because v_P, the standard deviation of the rate of return, is by definition never negative, while the Ω of a put is never positive. The analogy with a call might lead us to think that the Ω of a put must be less than or equal to -1 but this is not the case; it can be shown that the only restriction we can place on Ω is that it be less than or equal to zero. Consequently, it is possible for the volatility of a put to be less than the volatility of the stock.

Option Expected Return. To find the relationship between m_C and m_S, we need to go back to the derivation of the binomial formula. Recall that the equivalent portfolio has the same end-of-period values as the call for each possible outcome. That is,

$$uS\Delta + rB = C_u \qquad \text{and} \qquad dS\Delta + rB = C_d.$$

With Δ and B chosen in this way, we found that $C = S\Delta + B$. We can combine these in the following way:

$$uS\Delta - C_u = r(S\Delta - C) \quad \text{and} \quad dS\Delta - C_d = r(S\Delta - C).$$

Multiplying the first equation by q and the second by $1 - q$, then adding the respective left- and right-hand sides, gives

$$q[uS\Delta - C_u] + (1-q)[dS\Delta - C_d] = r(S\Delta - C).$$

By rearranging terms and substituting for m_S and m_C, we have

$$m_S S\Delta - m_C C = r(S\Delta - C).$$

Finally, rearranging again and using the definition of Ω, we obtain

$$m_C - r = \Omega(m_S - r).$$

That is, the *excess expected rate of return* (over the risk-free rate) *on the call is equal to Ω times the excess expected rate of return on the stock.* Since $\Omega \geq 1$, if the expected rate of return on the stock is greater (less) than the risk-free rate, then the expected rate of return on the call is never less (greater) than the expected rate of return on the stock.

The same relationship holds for puts:

$$m_P - r = \Omega(m_S - r).$$

Now, however, $\Omega \leq 0$. Consequently, we can only say that if the expected rate of return on the stock is greater (less) than the riskless rate, then the expected rate of return on the put is less (greater) than the riskless rate.

Option Beta. In all of this, m_S, the expected total return of the stock, could have been determined in any manner. However, if we also have a theory about how it is determined, we could then incorporate this theory into our results. The capital asset pricing model is an example of just such a theory. It says that, under certain conditions, the expected rate of return of a stock can be written in terms of the expected rate of return on a portfolio containing all available assets in proportion to their market values.[16] This portfolio is usually called the "market portfolio," and its expected total return will be denoted by m_M. Stating this relationship precisely,

$$m_S - r = \beta_S(m_M - r),$$

where β_S is the *beta* of the stock—that is, the covariance[17] of the stock's rate of return with that of the market portfolio, divided by the variance of the rate of return of the market portfolio. If we substitute this expression for $m_S - r$ into our earlier equation for $m_C - r$, we obtain

$$m_C - r = \Omega\beta_S(m_M - r).$$

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[16] For an introduction to the capital asset pricing model, see William F. Sharpe, *Investments*, 2nd Ed. (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1981), Chs. 5 and 6.

[17] If the market portfolio also followed a binomial process, with a total return over each period of u_M or d_M, then by definition this covariance would be

$$q_1(u - m_S)(u_M - m_M) + q_2(u - m_S)(d_M - m_M) + q_3(d - m_S)(u_M - m_M) + q_4(d - m_S)(d_M - m_M)$$

where q_1 is the probability that both the stock and the market will go up, q_2 is the probability that the stock will go up and the market will go down, and so on. Note, however, that the results in the text do not require that the *market portfolio* follow a binomial process.

It can be shown that $\Omega\beta_S$ is indeed the covariance of the rate of return on the call with the rate of return on the market divided by the variance of the rate of return on the market, so we can write the beta of a call β_C as

$$\beta_C = \Omega\beta_S.$$

The *option beta equals its elasticity times its underlying stock beta.* Since $\Omega \geq 1$ for a call, in the normal case of $\beta_S \geq 0$, we have $\beta_C \geq \beta_S$. For a put, $\Omega \leq 0$, so if $\beta_S \geq 0$, then $\beta_P \leq 0$. For both puts and calls, Ω will change from period to period due to stock price changes and the passage of time. Therefore, even if the beta of the stock remains constant, the beta of an option will not.

Theories like the capital asset pricing model, which seek to explain the relationships of rates of return on all assets, will imply a particular relationship between option and stock prices. The converse is not true, however. Indeed, we have seen that in deriving an option pricing formula we needed to know only some properties of the underlying stock. We did not need to know whether this stock was fairly priced relative to other stocks or, in fact, anything at all about other stocks. Although the option pricing formula and the capital asset pricing model may both be very useful, the validity of the former does not depend on the latter.[18]

Option Alpha. So far we have presumed that the option is properly valued relative to its associated stock, and that the stock is valued relative to the market portfolio according to the capital asset pricing model. However, suppose that independent estimates of m_C and m_S imply that our expected return and risk relationships are not satisfied. Then, if our predictions and models are correct, we will have isolated mispriced securities.

For example, suppose that our independent estimate of m_S implies that $m_S - r > \beta_S(m_M - r)$. We then believe the stock is underpriced and promises an expected return greater than that justified by its level of risk. This extra expected return is commonly termed the stock *alpha* and is determined by

$$m_S - r = \alpha_S + \beta_S(m_M - r).$$

Of course, α_S can be positive or negative, and is equal to zero only if we believe a stock is properly priced by the market.

[18] Indeed, in the limiting case discussed in the next section, the capital asset pricing model implies the option pricing formula, but the option pricing formula itself does not imply the capital asset pricing model. The option pricing theory is therefore more general than the capital asset pricing model. However, this should not be surprising, since the task of the option pricing formula, to explain the pricing relationship between particular contractually related securities, is clearly less ambitious than the task of the capital asset pricing model, which is to explain the pricing relationships among all securities.

The alpha of an option can be broken into two components: the associated stock alpha and the relative pricing relationship between the option and the stock. For a call, we can quantify the latter source by $\hat{\alpha}_C$ in

$$m_C - r = \hat{\alpha}_C + \Omega(m_S - r).$$

If the call is underpriced (overpriced) relative to the stock, then $\hat{\alpha}_C > 0$ (< 0). Putting the above two equations together gives

$$m_C - r = \hat{\alpha}_C + \Omega\alpha_S + \Omega\beta_S(m_M - r),$$

so that the call alpha[19] can be written as

$$\alpha_C = \hat{\alpha}_C + \Omega\alpha_S.$$

Since $\Omega \geq 1$, α_C will tend to be greater in magnitude than α_S unless this difference is fully offset by an opposing relative mispricing between the call and the stock.

For a put, we have a similar relationship, $m_P - r = \hat{\alpha}_P + \Omega\alpha_S + \Omega\beta_S(m_M - r)$, but some of the conclusions are different. Since $\Omega \leq 0$, if $\hat{\alpha}_P = 0$, then α_P and α_S have opposite signs. In other words, if a put is properly priced relative to the stock, and the stock is underpriced (overpriced) relative to the market, then the put must be overpriced (underpriced) relative to the market.

Figure 5-1(a) illustrates the relationship between expected rate of return and beta given by the capital asset pricing model for some representative options that are properly priced relative to the underlying stock. If an option is underpriced (overpriced) relative to the stock, then it will lie above (below) the line shown. If the stock itself is properly priced relative to the market, then all other properly priced securities will lie somewhere along the line shown. If instead the stock is underpriced (overpriced) relative to the market, then all properly priced securities will lie along another straight line which crosses the vertical axis at the same point as the line shown but has a smaller (larger) slope.

Figure 5-1(b) shows the corresponding relationship between expected rate of return and volatility for properly priced options. Note that along the lower section of the graph, expected rate of return decreases as volatility increases. Even though a put has a higher variability of return than a default-free bond, it has a lower expected rate of return. However, this is not surprising. We know from Section 2-3 that a put will be equivalent to a

[19] The alpha of either the stock or the option will depend on the length of time before equilibrium is restored. Other things equal, an alpha will be greater the shorter this time period.

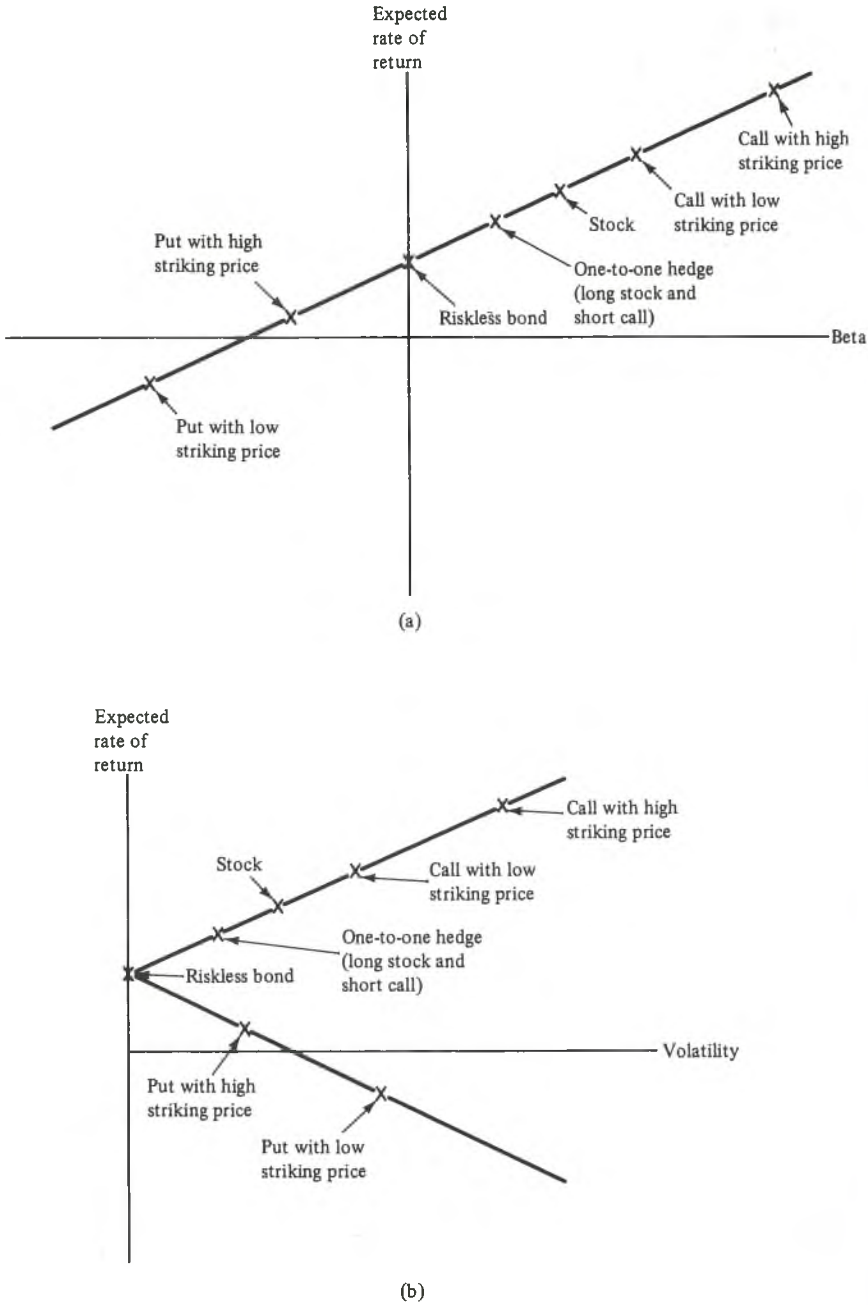

Figure 5-1 Relationship of Expected Rate of Return to Beta and Volatility

portfolio containing a long position in default-free bonds and a short position in the stock. If the expected rate of return on the stock is greater than the riskless rate, then the expected rate of return on such a portfolio must be less than the riskless rate. Although this conclusion does not depend on the validity of the capital asset pricing model, the two are completely consistent, since we know that a put has a negative beta.

Risk and Expected Return in Terms of Dollar Changes. Sometimes it is convenient to express risk and expected return in terms of the dollar change, which is the rate of return multiplied by the price at the beginning of the period. Since this price is not a random variable, we are in effect simply multiplying the random rate of return by a constant. Hence, for the stock, the expected dollar change will be $(m_S - 1)S$ and the standard deviation of the dollar change will be $v_S S$. For the call, the corresponding values will be $(m_C - 1)C$ and $v_C C$.

We can now use our previous results to write

$$v_C C = v_S S\Delta,$$
$$m_C C - rC = [m_S S - rS]\Delta.$$

In other words, Δ tells us:

1. The ratio of the standard deviation of the dollar change in the call value to the standard deviation of the dollar change in the stock price.
2. The ratio of the excess expected dollar change (over the risk-free dollar change) in the call price to the excess expected dollar change in the stock price.

If we can show that $\Delta \leq 1$, then we can also make the following statements:

1. In absolute terms (standard deviation of dollar changes), the call is never more risky than the stock.
2. If the expected dollar change in the stock is nonnegative, then the expected dollar changes on the call are never greater than the expected dollar changes on the stock.

To see that $\Delta \leq 1$, consider the following argument. $\Delta \leq 1$ if and only if $C_u - C_d \leq (u - d)S$, or $uS - C_u \geq dS - C_d$. From our earlier development, we know that this can be rewritten as

$$r^{-(n-1)}E\{uSu^j d^{n-1-j} - \max[0, uSu^j d^{n-1-j} - K]\}$$
$$\geq r^{-(n-1)}E\{dSu^j d^{n-1-j} - \max[0, dSu^j d^{n-1-j} - K]\},$$

or

$$E \min[uSu^j d^{n-1-j}, K] \geq E \min[dSu^j d^{n-1-j}, K].$$

The last inequality obviously holds, so Δ is indeed less than or equal to one.

The corresponding results for a put are:

$$v_P P = -v_S S\Delta$$
$$m_P P - rP = [m_S S - rS]\Delta.$$

The interpretation of Δ in terms of information about dollar changes is similar to that for a call. Also, it can be shown that $\Delta \geq -1$, so we can conclude that in absolute terms (standard deviation of dollar changes), a put is never more risky than the stock.

Risk and Expected Return Over Many Periods. We have found the risk and expected return of a call relative to that of the stock over a single period. The relationships were simple and direct. This is just as we would have expected, since we had earlier shown that over any single period the call was equivalent to a particular levered long position in the stock.

Often, we will also want to know about the corresponding measures for positions that are held over a number of periods. The definitions of risk and expected return can be easily extended for the stock. If there are no payouts, we simply substitute the value of the security at the end of the entire holding interval in place of its value at the end of the current period. Straightforward calculations show that over k periods, the expected total return, $m_S(k)$, and variance of total return, $v_S^2(k)$, are

$$m_S(k) = [qu + (1-q)d]^k = m_S^k,$$
$$v_S^2(k) = [qu^2 + (1-q)d^2]^k - [qu + (1-q)d]^{2k}.$$

We might hope that there will again be a simple way to relate the risk and expected return of the option to that of the stock. However, this is not the case. The reason is that while the call is equivalent to a portfolio of stock and bonds that is readjusted every period in a specified way, it is *not* equivalent to *any* portfolio of stock and bonds whose proportions remain fixed over the entire interval. It is easy to see why this is true. At the end of k periods, the stock can take on $k+1$ possible values. So can the call. With only two choice variables—the amount of stock and the amount of bonds—we could not hope to find a portfolio whose end-of-interval value would be the same as that of the call in each of the $k+1$ possible outcomes. In other words, over more than one period, the call offers a pattern of returns that cannot be duplicated by any fixed portfolio of stock and bonds. Hence, its risk and expected return cannot be expressed in terms of those of such a portfolio.

Nevertheless, we have at hand all of the information we need. For any future date, we know the value of a call as a function of the stock price at that time. We also know the probability distribution of this future stock price. We can thus in principle calculate the risk and expected return—or any other statistic we might want—for the option over any interval, but the exact results will usually be quite complicated. We will pursue this further in Section 6-5.

If the end of the holding interval is the expiration date, then things will be somewhat simpler. For example, the expected total return on a call will then be the expected total return on the stock times an adjustment factor. This factor is the current value that the call would have if $r = m_s$, divided by the current value of the call.

A Numerical Example. We can illustrate all of this with the numerical example developed in Section 5-4. There we assumed that $S = 80$, $K = 80$, $n = 3$, $u = 1.5$, $d = .5$, and $r = 1.1$. The resulting call values were $C = 34.065$, $C_u = 60.463$, $C_d = 2.974$, $C_{uu} = 107.272$, $C_{du} = 5.454$, and $C_{dd} = 0$. Since we now need to know the actual distribution of the stock price, let us suppose that $q = .7$.

We can calculate the initial values of m_S, v_S, m_C, and v_C directly from their definitions. These values are

$$\begin{aligned}
m_S &= .7(1.5) + .3(.5) = 1.2,\\
m_S - r &= 1.2 - 1.1 = .1,\\
v_S^2 &= .7(1.5 - 1.2)^2 + .3(.5 - 1.2)^2 = .21,\\
v_S &= .458,\\
m_C &= .7\left(\frac{60.463}{34.065}\right) + .3\left(\frac{2.974}{34.065}\right) = .7(1.775) + .3(.087) = 1.269,\\
m_C - r &= .169,\\
v_C^2 &= .7(1.775 - 1.269)^2 + .3(.087 - 1.269)^2 = .598,\\
v_C &= .773.
\end{aligned}$$

To verify that these values are consistent with our formulas, we first need to evaluate Ω. This gives

$$\Omega = \frac{S\Delta}{C} = \frac{C_u - C_d}{(u-d)C} = \frac{60.463 - 2.974}{34.065} = 1.689.$$

To confirm the formulas, we note that

$$\begin{aligned}
v_C &= \Omega v_S = 1.689(.458) = .773,\\
m_C - r &= \Omega(m_S - r) = 1.689(.1) = .169.
\end{aligned}$$

If we wished to compute the expected total return over, for instance, two periods, we would find

$$m_S(2) = (1.2)^2 = 1.44,$$
$$m_C(2) = (.7)^2\left(\frac{107.272}{34.065}\right) + 2(.7)(.3)\left(\frac{5.454}{34.065}\right) + (.3)^2\left(\frac{0}{34.065}\right) = 1.61.$$

Some Additional Comments. A few other points are worth emphasizing. Although the value of the call, in terms of the stock (our option pricing formula) did not depend on q, the expected rate of return of the call certainly does depend on q, through its dependence on m_S, the expected rate of return of the stock. The higher the probability of an up movement in the stock, the higher its expected rate of return, and the higher the expected rate of return of the option, just as we would have thought.

Furthermore, the risk and expected rate of return on the call were that which would hold if the call were in equilibrium at the beginning and end of the period. If the call price is currently out of equilibrium, but will move back into line at the end of some interval (possibly one period), then we can calculate the risk and expected return over this interval by substituting the current market price in place of the current formula value, C. If the call price could move even more out of equilibrium, then the risk and expected return on holding a call over the interval could conceivably be almost anything. This squares with our earlier observation that if an arbitrage strategy is liquidated before expiration, it will not necessarily be riskless. We could then make definite statements about risk and expected return over any holding period only if we make some assumption about the disequilibrium behavior of option market prices. However, we know that an investment in an undervalued call will, if held until expiration, have a higher expected rate of return than its equivalent portfolio of stock and bonds.

Finally, all of the results in this section hold for an American option on a stock that pays dividends. There are only two minor differences. First, the total return on holding the stock should include reinvestment of cash dividends. Second, the risk and expected return relationships for holding the option over the next period will not apply if the option should be exercised immediately.

5-6. THE BLACK-SCHOLES FORMULA

The Effect of More Frequent Trading. In reading the previous sections, there is a natural tendency to associate with each period some particular length of calendar time, perhaps a day. With this in mind, you may have had two objections. In the first place, prices a day from now may take on many more than just two possible values. Furthermore, the market is not open for trading only once a day, but, instead, trading takes place almost continuously.

These objections are certainly valid. Fortunately, our option pricing approach has the flexibility to meet them. Although it might have been natural to think of a period as one day, there was nothing that forced us to do so. We could have taken it to be a much shorter interval—say an hour—or even a minute. By doing so we have met both objections simultaneously. Trading would take place far more frequently, and the stock price could take on hundreds of values by the end of the day.

However, if we do this, we have to make some other adjustments to keep the probability small that the stock price will change by a large amount over a minute. We do not want the stock to have the same percentage up and down moves for one minute as it did before for one day. But again there is no need for us to have to use the same values. We could, for example, think of the price as making only a very small percentage change over each minute.

To make this more precise, suppose that h represents the elapsed time between successive stock price changes. That is, if t is the fixed length of calendar time to expiration, and n is the number of periods of length h prior to expiration, then

$$h \equiv t/n.$$

As trading takes place more and more frequently, h gets closer and closer to zero. We must then adjust the interval-dependent variables r, u, and d in such a way that we obtain empirically realistic results as h becomes smaller, or, equivalently, as $n \to \infty$.

When we were thinking of the periods as having a fixed length, r represented both the interest rate over a fixed length of calendar time and the interest rate over one period. Now we need to make a distinction between these two meanings. We will let r continue to mean one plus the interest rate over a fixed length of calendar time. When we have occasion to refer to one plus the interest rate over a period (trading interval) of length h, we will use the symbol $\hat{r}$.

Clearly, the size of $\hat{r}$ depends on the number of subintervals, n, into which t is divided. Over the n periods until expiration, the total return is $\hat{r}^n$, where $n = t/h$. Now not only do we want $\hat{r}$ to depend on n, but we want it to depend on n in a particular way—so that as n changes the total return $\hat{r}^n$ over the fixed time t remains the same. This is because the interest rate obtainable over some fixed length of calendar time should have nothing to do with how we choose to think of the length of the time interval h.

If r (without the "hat") denotes one plus the rate of interest over a *fixed* unit of calendar time, then over elapsed time t, r^t is the total return.[20] Observe that this measure of total return does not depend on n. As we have argued, we want to choose the dependence of $\hat{r}$ on n, so that

$$\hat{r}^n = r^t,$$

for any choice of n. Therefore, $\hat{r} = r^{t/n}$. This last equation shows how $\hat{r}$ must depend on n for the total return over elapsed time t to be independent of n.

We also need to define u and d in terms of n. At this point, there are two significantly different paths we can take. Depending on the definitions we choose, as $n \to \infty$ (or, equivalently, as $h \to 0$), we can have either a continuous or a jump stochastic process. In the first situation, very small random changes in the stock price will be occurring in each very small time interval. The stock price will fluctuate incessantly, but its path can be drawn without lifting pen from paper. In contrast, in the second case, the stock price will usually move in a smooth deterministic way, but will occasionally experience sudden discontinuous changes. Both can be derived from our binomial process simply by choosing how u and d depend on n. In this chapter, we examine only the continuous process which leads to the option pricing formula originally derived by Fischer Black and Myron Scholes. We will postpone discussion of the jump process formula until Chapter 7.

Recall that we supposed that over each period the stock price would experience a one plus rate of return of u with probability q and d with probability $1 - q$. It will be easier and clearer to work, instead, with the natural logarithms of the one plus rate of return, log u and log d. This gives the continuously compounded rate of return on the stock over each period.[21] It is a random variable which, in each period, will be equal to log u with probability q and log d with probability $1 - q$.

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[20] The scale of this unit (perhaps a day, or a year) is unimportant as long as r and t are expressed in the same scale.

[21] Continuously compounded rates of interest are commonly used by banks on savings accounts. To convert the discrete one plus rate of return u over a single period into a continuously compounded rate, consider what happens as we divide the period into m subperiods of equal length. Suppose we denote by g/m the rate of increase required over each of these subperiods to produce u over the entire period; that is,

$$u = (1 + g/m)^m.$$

g itself depends on m and is the m subperiod compounded rate of increase. Now, as $m \to \infty$, g becomes the continuously compounded rate of increase. In the limit, it can be shown that $u = e^g$, where e is the exponential constant, $e = 2.718 \ldots$, which is the limiting value of $[1 + (1/k)]^k$ as $k \to \infty$. Therefore, the continuously compounded rate of increase $g = \log u$. Similarly, since $r^t = e^{(\log r)t}$, the continuously compounded rate of interest is log r.

Consider a typical sequence of five moves, say u, d, u, u, d. Then, $S^* = uduudS$, $S^*/S = u^3d^2$, and $\log(S^*/S) = 3 \log u + 2 \log d$. More generally, over n periods,

$$\log(S^*/S) = j \log u + (n - j) \log d = j \log(u/d) + n \log d,$$

where j is the (random) number of upward moves occurring during the n periods to expiration. Therefore, the expected value of $\log(S^*/S)$ is

$$E[\log(S^*/S)] = E(j)[\log(u/d)] + n \log d$$

and its variance is

$$\mathrm{Var}[\log(S^*/S)] = \mathrm{Var}(j)[\log(u/d)]^2.$$

Each of the n possible upward moves has probability q. Thus, $E(j) = nq$. Also, since the variance each period is

$$q(1 - q)^2 + (1 - q)(0 - q)^2 = q(1 - q),$$

then $\mathrm{Var}(j) = nq(1 - q)$. Combining all of this, we have

$$\begin{aligned} E[\log(S^*/S)] &= [q \log(u/d) + \log d]n \equiv \hat{\mu}n \\ \mathrm{Var}[\log(S^*/S)] &= q(1 - q)[\log(u/d)]^2 n \equiv \hat{\sigma}^2 n. \end{aligned}$$

Let us go back to our discussion. We were considering dividing up our original longer time period (a day) into many shorter periods (a minute or even less). Over a fixed length of calendar time t, our procedure calls for making n larger and larger. Now if we held everything else constant while we let n become large, we would be faced with the problem we talked about earlier. In fact, we would certainly not reach a reasonable conclusion if either $\hat{\mu}n$ or $\hat{\sigma}^2 n$ went to zero or infinity as n became large. Since t is a fixed length of time, in searching for a realistic result, we must make the appropriate adjustments in u, d, and q. In doing that, we would at least want the mean and variance of the continuously compounded rate of return of the assumed stock price movement to coincide with that of the actual stock price as $n \rightarrow \infty$. Suppose we label the actual empirical values of $\hat{\mu}n$ *and* $\hat{\sigma}^2 n$ as μt and $\sigma^2 t$, respectively. Then we would want to choose u, d, and q, so that

$$\left.\begin{aligned} [q \log(u/d) + \log d]n &\rightarrow \mu t \\ q(1 - q)[\log(u/d)]^2 n &\rightarrow \sigma^2 t \end{aligned}\right\} \quad \text{as} \quad n \rightarrow \infty.$$

A little algebra shows we can accomplish this by letting

$$u = e^{\sigma\sqrt{t/n}}, \qquad d = e^{-\sigma\sqrt{t/n}}, \qquad \text{and} \qquad q = \tfrac{1}{2} + \tfrac{1}{2}(\mu/\sigma)\sqrt{t/n}.$$

In this case, for any n,

$$\hat{\mu}n = \mu t \qquad \text{and} \qquad \hat{\sigma}^2 n = [\sigma^2 - \mu^2(t/n)]t.$$

Clearly, as $n \to \infty$, $\hat{\sigma}^2 n \to \sigma^2 t$, while $\hat{\mu}n = \mu t$ for all values of n.

Alternatively, we could have chosen u, d, and q so that the mean and variance of the future stock price for the discrete binomial process approach the prespecified mean and variance of the actual stock price as $n \to \infty$. However, just as we would expect, the same values will accomplish this as well. Since this would not change our conclusions, and it is computationally more convenient to work with the continuously compounded rates of return, we will proceed in that way.

This satisfies our initial requirement that the limiting means and variances coincide, but we still need to verify that we are arriving at a sensible limiting probability distribution of the continuously compounded rate of return. The mean and variance only describe certain aspects of that distribution.

For our model, the random continuously compounded rate of return over a period of length t is the sum of n independent random variables, each of which can take the value log u with probability q and log d with probability $1 - q$. We wish to know about the distribution of this sum as n becomes large and q, u, and d are chosen in the way described. We need to remember that as we change n, we are not simply adding one more random variable to the previous sum, but instead are changing the probabilities and possible outcomes for every member of the sum. At this point, we can rely on a form of the central limit theorem which, when applied to our problem, says that as $n \to \infty$, if

$$\frac{q\,|\log u - \hat{\mu}|^3 + (1-q)\,|\log d - \hat{\mu}|^3}{\hat{\sigma}^3\sqrt{n}} \to 0,$$

then

$$\text{Prob}\left\{\left[\frac{\log(S^*/S) - \hat{\mu}n}{\hat{\sigma}\sqrt{n}}\right] \le z\right\} \to N(z),$$

where $N(z)$ is the standard normal distribution function. Putting this into words, as the number of periods into which the fixed length of time to expiration is divided approaches infinity, the probability that the standardized continuously compounded rate of return of the stock through the expiration date is not greater than the number z approaches the probability under a standard normal distribution.

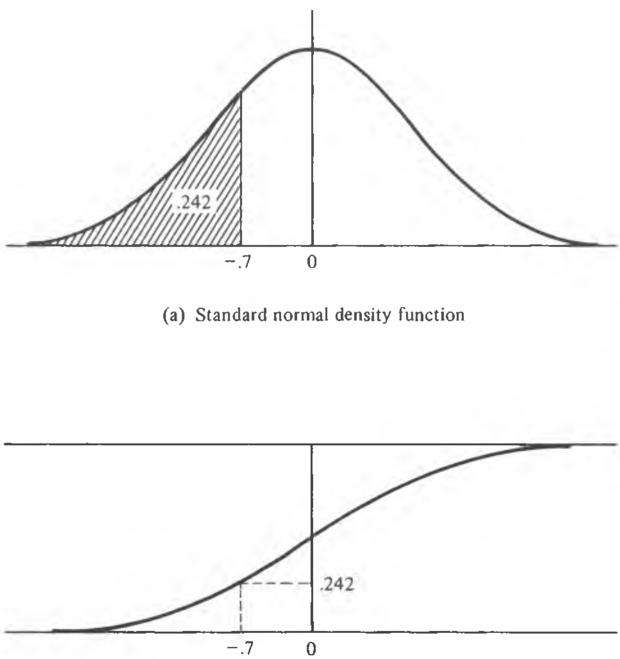
The initial condition says roughly that higher-order properties of the distribution, such as how it is skewed, become less and less important, relative to its standard deviation, as $n \to \infty$. We can verify that the condition is satisfied by making the appropriate substitutions and finding

$$\frac{q\,|\log u - \hat{\mu}|^3 + (1-q)\,|\log d - \hat{\mu}|^3}{\hat{\sigma}^3\sqrt{n}} = \frac{(1-q)^2 + q^2}{\sqrt{nq(1-q)}},$$

$$\hat{\sigma}^3\sqrt{n}$$

which goes to zero as $n \to \infty$ since $q = \frac{1}{2} + \frac{1}{2}(\mu/\sigma)\sqrt{t/n}$.

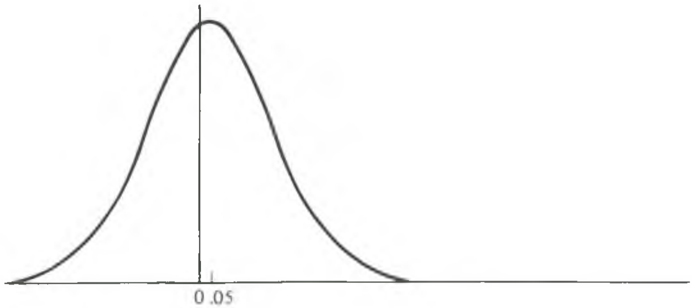
Properties of Normal and Lognormal Random Variables. Since the normal and lognormal distributions are important to our analysis of options, it will be useful to review their properties. The density function of a normally distributed random variable, depicted in Figure 5-2(c), is described by a "bell-shaped" curve, familiar from almost all elementary books on statistics.



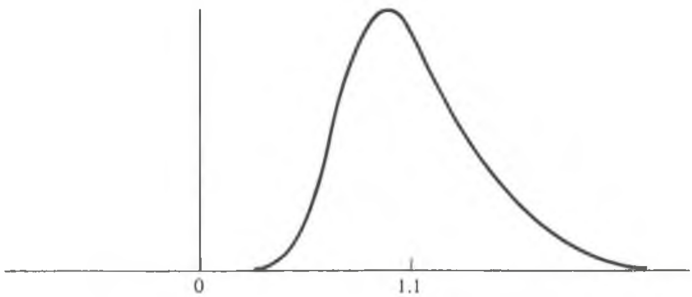
(a) Standard normal density function

(b) Standard normal distribution function

Figure 5-2 Comparison of Normal and Lognormal Probability Functions



(c) Normal density function
(Mean: .05, Variance: .09)



(d) Lognormal density function
(Mean: 1.100, Variance: .114)

Figure 5-2 Comparison of Normal and Lognormal Probability Functions (continued)

It is symmetrical about the mean, and the mean, median, and mode are all equal. About two-thirds of the area under the curve lies within one standard deviation of the mean. Furthermore, the distribution is completely specified by its mean and standard deviation.

A standardized normally distributed random variable has a mean of zero and a standard deviation of one. The standard normal density function,

$$N'(x) \equiv (1/\sqrt{2\pi})e^{-x^2/2},$$

is shown in Figure 5-2(a). The standard normal distribution function, $N(z)$, gives the area under this density from $-\infty$ to z. That is, it gives the probability that the random variable will take on a value less than or equal to z. Therefore,

$$N(-\infty) = 0, \qquad 0 \leq N(z) \leq 1, \qquad \text{and} \qquad N(+\infty) = 1.$$

Furthermore, from the symmetry of the distribution, $N(-z) = 1 - N(z)$. Figure 5-2(a, b) illustrates some of these properties for $z = -.7$.

In our case, it is the variable

$$\frac{\log(S^*/S) - \mu t}{\sigma\sqrt{t}}$$

which has a standardized normal distribution. Consequently, $\log(S^*/S)$ has a normal distribution with mean μt and variance $\sigma^2 t$, and $\log S^*$ has a normal distribution with mean $\mu t + \log S$ and variance $\sigma^2 t$. This is shown in Figure 5-2(c) for $\mu t = .05$ and $\sigma^2 t = .09$. Since it is customary to think in terms of the price relative S^*/S, Figure 5-2(d) shows the implied shape of the density of $x \equiv S^*/S$, given that $\log(S^*/S)$ is normally distributed with mean $\mu t = .05$ and variance $\sigma^2 t = .09$. This distribution is termed lognormal. That is, whenever the random variable $\log x$ is normally distributed, then x itself is lognormally distributed. To clarify this transformation, remember that

$$\log x = \begin{cases} -\infty \\ 0 \\ 1 \\ +\infty \end{cases} \rightarrow \quad x = \begin{cases} 0 \\ 1 \\ e \\ +\infty \end{cases}$$

These properties are illustrated in Figure 5-3, which shows a graph of the log function.

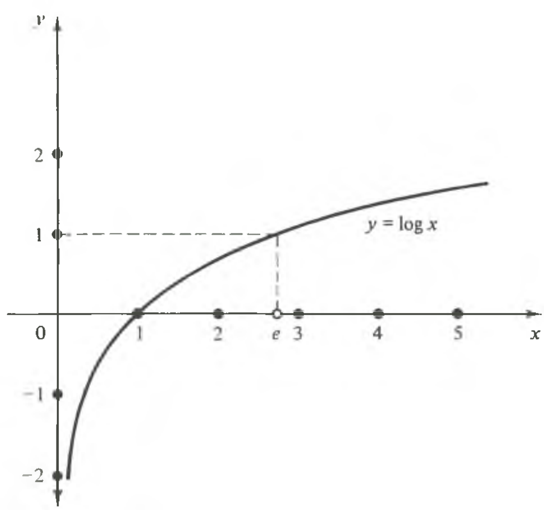


Figure 5-3 The Natural Logarithm

While $\log(S^*/S)$ is symmetric, S^*/S is skewed to the right, and the probability that $S^*/S \leq 0$ is zero. While the mean and variance of $\log(S^*/S)$ are μt and $\sigma^2 t$, the mean and variance of (S^*/S) are

$$e^{\mu t + \sigma^2 t/2} \qquad \text{and} \qquad e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1).$$

For $\mu t = .05$ and $\sigma^2 t = .09$, the mean is 1.100 and the variance is .114. The median of (S^*/S) is $e^{\mu t}$ and its mode is $e^{\mu t - \sigma^2 t}$. If we were interested in the corresponding quantities for S^* rather than (S^*/S), we would simply multiply the mean, mode, and median by S and the variance by S^2.

We have shown that the multiplicative binomial model includes the lognormal distribution as a limiting case. This distribution has a number of sensible properties. It implies that stocks have limited liability, since, provided $S > 0$, S^* can never become negative. On the other hand, there is no upper limit on how far the stock price might rise, but very large increases are quite unlikely. Because of the symmetry of the normal distribution, equal up and down movements in $\log(S^*/S)$ about its mean are equally likely. For example, if $S = \$100$ and $E(S^*) = \$100$, then an increase to $S^* = \$133$ is just as likely as a decrease to $S^* = \$75$. This follows since $\log(133/100) = -\log(75/100)$. In other words, equal *relative* changes in S^* about its mean are equally likely. We might compare this with the supposition that equal *absolute* changes in S^* about the mean are equally likely. In this latter case, an increase to $S^* = \$133$ would be just as likely as a decrease to $S^* = \$67$. Taken in the extreme, the absolute hypothesis would imply moves to $S^* = \$0$ and $S^* = \$200$ were equally likely. To be sure, for most stocks the empirical reality may lie somewhere between the relative and absolute hypotheses, and we will consider other possibilities in Chapter 7.

Convergence to the Black-Scholes Option Pricing Formula. Black and Scholes began directly with continuous trading and the assumption of a lognormal distribution for stock prices. Their approach relied on some quite advanced mathematics. However, since our approach contains continuous trading and the lognormal distribution as a limiting case, the two resulting formulas should then coincide. We will see shortly that this is indeed true, and we will have the advantage of using a much simpler method. It is important to remember, however, that the economic arguments we used to link the option value and the stock price are the same as those advanced by Black and Scholes and Merton.

The formula derived by Black and Scholes, rewritten in terms of our notation, is

BLACK-SCHOLES OPTION PRICING FORMULA

$$C = SN(x) - Kr^{-t}N(x - \sigma\sqrt{t})$$

where

$$x \equiv \frac{\log(S/Kr^{-t})}{\sigma\sqrt{t}} + \tfrac{1}{2}\sigma\sqrt{t}$$

Here is one interpretation of the formula. If we exercise the call on the expiration date we will receive the stock, but in return we will have to pay the striking price. Of course, this exchange will not take place unless the call finishes in-the-money. The first term in the formula, $SN(x)$, is the present value of receiving the stock if and only if $S^* > K$, and the second term, $-Kr^{-t}N(x - \sigma\sqrt{t})$, is the present value of paying the striking price if and only if $S^* > K$. Just as we would expect from this interpretation, if S is very large relative to K, then $N(x) \approx N(x - \sigma\sqrt{t}) \approx 1$, and $C \approx S - Kr^{-t}$.

The formula has another interpretation which is particularly useful. Shortly, we will show that $N(x) = \Delta$, the number of shares in the equivalent portfolio. We know from our earlier discussions that $C = S\Delta + B$, where B is the dollar amount invested in default-free bonds in the equivalent portfolio. We can then see that $B = -Kr^{-t}N(x - \sigma\sqrt{t})$ directly from inspection of the Black-Scholes formula. As we stated in Chapter 2, the equivalent portfolio consists of a long position in less than one share of stock financed partly by borrowing. *The first term in the Black-Scholes formula, $SN(x)$, is the amount invested in the stock; the second term, $Kr^{-t}N(x - \sigma\sqrt{t})$, is the amount borrowed.*

We now wish to confirm that our binomial formula converges to the Black-Scholes formula when t is divided into more and more subintervals, and $\hat{r}$, u, d, and q are chosen in the way we described—that is, in a way such that the multiplicative binomial probability distribution of stock prices goes to the lognormal distribution.

For easy reference, let us recall our binomial option pricing formula:

$$C = S\Phi[a; n, p'] - K\hat{r}^{-n}\Phi[a; n, p].$$

The similarities are readily apparent. $\hat{r}^{-n}$ is, of course, always equal to r^{-t}. Therefore, to show the two formulas converge, we need only show that as $n \to \infty$,

$$\Phi[a; n, p'] \to N(x) \qquad \text{and} \qquad \Phi[a; n, p] \to N(x - \sigma\sqrt{t}).$$

We will consider only $\Phi[a; n, p]$, since the argument is exactly the same for $\Phi[a; n, p']$.

The complementary binomial distribution function $\Phi[a; n, p]$ is the probability that the sum of n random variables, each of which can take on the value 1 with probability p and 0 with probability $1 - p$, will be greater than or equal to a. We know that the random value of this sum, j, has mean np and standard deviation $\sqrt{np(1-p)}$. Therefore,

$$1 - \Phi[a; n, p] = \text{Prob}[j \le a - 1] = \text{Prob}\left[\frac{j - np}{\sqrt{np(1-p)}} \le \frac{a - 1 - np}{\sqrt{np(1-p)}}\right].$$

Now we can make an analogy with our earlier discussion. If we consider a stock which in each period will move to uS with probability p and dS with probability $1 - p$, then $\log(S^*/S) = j \log(u/d) + n \log d$. The mean and variance of the continuously compounded rate of return of this stock are

$$\hat{\mu}_p = p \log(u/d) + \log d \qquad \text{and} \qquad \hat{\sigma}_p^2 = p(1-p)[\log(u/d)]^2.$$

Using these equalities, we find that

$$\frac{j - np}{\sqrt{np(1-p)}} = \frac{\log(S^*/S) - \hat{\mu}_p n}{\hat{\sigma}_p \sqrt{n}}.$$

Recall from the binomial formula that

$$a - 1 = \log(K/Sd^n)/\log(u/d) - \epsilon = [\log(K/S) - n \log d]/\log(u/d) - \epsilon,$$

where ϵ is a number between zero and one. Using this and the definitions of $\hat{\mu}_p$ and $\hat{\sigma}_p^2$, with a little algebra, we have:

$$\frac{a - 1 - np}{\sqrt{np(1-p)}} = \frac{\log(K/S) - \hat{\mu}_p n - \epsilon \log(u/d)}{\hat{\sigma}_p \sqrt{n}}.$$

Putting these results together,

$$1 - \Phi[a; n, p] = \text{Prob}\left[\frac{\log(S^*/S) - \hat{\mu}_p n}{\hat{\sigma}_p \sqrt{n}} \le \frac{\log(K/S) - \hat{\mu}_p n - \epsilon \log(u/d)}{\hat{\sigma}_p \sqrt{n}}\right].$$

We are now in a position to apply the central limit theorem. First, we must check if the initial condition,

$$\frac{p\,|\log u - \hat{\mu}_p|^3 + (1-p)\,|\log d - \hat{\mu}_p|^3}{\hat{\sigma}_p \sqrt{n}} = \frac{(1-p)^2 + p^2}{\sqrt{np(1-p)}} \to 0$$

as $n \to \infty$, is satisfied. By first recalling that $p \equiv (\hat{r} - d)/(u - d)$, and then $\hat{r} = r^{t/n}$, $u = e^{\sigma\sqrt{t/n}}$, and $d = e^{-\sigma\sqrt{t/n}}$, it is possible to show that as $n \to \infty$,

$$p \to \frac{1}{2} + \frac{1}{2}\left(\frac{\log r - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\frac{t}{n}}.$$

As a result, the initial condition holds, and we are justified in applying the central limit theorem.

To do so, we need only evaluate $\hat{\mu}_p n$, $\hat{\sigma}_p^2 n$, and $\log(u/d)$ as $n \to \infty$.[22] Examination of our discussion for parameterizing q shows that as $n \to \infty$,

$$\hat{\mu}_p n \to (\log r - \tfrac{1}{2}\sigma^2)t \qquad \text{and} \qquad \hat{\sigma}_p\sqrt{n} \to \sigma\sqrt{t}.$$

Furthermore, $\log(u/d) \to 0$ as $n \to \infty$.

For this application of the central limit theorem, then, since

$$\frac{\log(K/S) - \hat{\mu}_p n - \epsilon \log(u/d)}{\hat{\sigma}_p\sqrt{n}} \to z \equiv \frac{\log(K/S) - (\log r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},$$

we have

$$1 - \Phi[a; n, p] \to N(z) = N\left[\frac{\log(Kr^{-t}/S)}{\sigma\sqrt{t}} + \tfrac{1}{2}\sigma\sqrt{t}\right].$$

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[22] A surprising feature of this evaluation is that although $p \neq q$ and thus $\hat{\mu}_p \neq \hat{\mu}$, and $\hat{\sigma}_p \neq \hat{\sigma}$, nonetheless $\hat{\sigma}_p\sqrt{n}$ and $\hat{\sigma}\sqrt{n}$ have the same limiting value as $n \to \infty$. By contrast, since $\mu \neq \log r - \frac{1}{2}\sigma^2$, $\hat{\mu}_p n$ and $\hat{\mu} n$ do not. This results from the way we needed to specify u and d to obtain convergence to a lognormal distribution. Rewriting this as $\sigma\sqrt{t} = (\log u)\sqrt{n}$, it is clear that the limiting value σ of the standard deviation does not depend on p or q, and hence must be the same for either. However, at any point before the limit, since

$$\hat{\sigma}^2 n = [\sigma^2 - \mu^2(t/n)]t \qquad \text{and} \qquad \hat{\sigma}_p^2 n = [\sigma^2 - (\log r - \tfrac{1}{2}\sigma^2)^2(t/n)]t,$$

$\hat{\sigma}$ and $\hat{\sigma}_p$ will generally have different values.

The fact that $\hat{\mu}_p n \to (\log r - \frac{1}{2}\sigma^2)t$ can also be derived from the property of the lognormal distribution that

$$\log E[S^*/S] = \mu_p t + \tfrac{1}{2}\sigma^2 t,$$

where E and μ_p are measured with respect to probability p. Since $p = (\hat{r} - d)/(u - d)$, it follows that $\hat{r} = pu + (1 - p)d$. For independently distributed random variables, the expectation of a product equals the product of their expectations. Therefore,

$$E[S^*/S] = [pu + (1 - p)d]^n = \hat{r}^n = r^t.$$

Substituting r^t for $E[S^*/S]$ in the previous equation, we have

$$\mu_p = \log r - \tfrac{1}{2}\sigma^2.$$

The final step in the argument is to use the symmetry property of the standard normal distribution that $1 - N(z) = N(-z)$. Therefore, as $n \to \infty$,

$$\Phi[a; n, p] \to N(-z) = N\left[\frac{\log(S/Kr^{-t})}{\sigma\sqrt{t}} - \tfrac{1}{2}\sigma\sqrt{t}\right] = N(x - \sigma\sqrt{t}).$$

Since a similar argument holds for $\Phi[a; n, p']$, this completes our demonstration that the binomial option pricing formula contains the Black-Scholes formula as a limiting case.[23]

The Continuous-Trading Valuation Equation.[24] When Black and Scholes originally derived their formula, they followed a different line of argument. However, we can use our simpler binomial model to explain their approach. In our original binomial development, recall that our ability to create an equivalent portfolio led to the following equation (somewhat rewritten)

$$\left[\frac{\hat{r} - d}{u - d}\right]C_u + \left[\frac{u - \hat{r}}{u - d}\right]C_d - \hat{r}C = 0 \tag{7}$$

relating the value of a call at the beginning of any period to its possible values at the end of the period. For our current purposes, it will be more convenient to write C as $C(S, t)$, C_u as $C(uS, t - h)$, and C_d as $C(dS, t - h)$.

By their more difficult methods, Black and Scholes obtained directly a partial differential equation analogous to our discrete-time difference equation. Their equation is

$$\tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (\log r)S\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - (\log r)C = 0, \tag{8}$$

where $\partial^2 C/\partial S^2$, $\partial C/\partial S$, and $\partial C/\partial t$ are partial derivatives, and $\log r$ is the continuously compounded rate of interest. The value C of the call was then derived by solving this equation.

Based on our previous analysis, we would now suspect that, as we divide up the time to expiration into more and more intervals with smaller and smaller moves in the way described earlier, our binomial valuation equation would approach the continuous-time valuation equation of Black and Scholes. We will turn now to an intuitive confirmation of this.

---

[23] The only difference is that as $n \to \infty$,

$$p' \to \frac{1}{2} + \frac{1}{2}\left(\frac{\log r + \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\frac{t}{n}}.$$

[24] The material in this subsection will be used only in Section 7-8.

If we choose $\hat{r}$, u, and d in the way described earlier, and substitute these values into our binomial valuation equation, we obtain

$$\left[\frac{r^h - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}\right] C(e^{\sigma\sqrt{h}}S, t - h) + \left[\frac{e^{\sigma\sqrt{h}} - r^h}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}\right] C(e^{-\sigma\sqrt{h}}S, t - h) - r^h C(S, t) = 0.$$

Now let us express $C(e^{\sigma\sqrt{h}}S, t - h)$ and $C(e^{-\sigma\sqrt{h}}S, t - h)$ as a Taylor series around the point (S, t). We will be interested only in terms multiplied by $\sqrt{h}$ or by h, since the remaining terms will become negligible, relative to these, as h becomes small. For an up move, we have

$$C(e^{\sigma\sqrt{h}}S, t - h) = C(S, t) + (e^{\sigma\sqrt{h}} - 1)S\frac{\partial C}{\partial S} + \tfrac{1}{2}(e^{\sigma\sqrt{h}} - 1)^2\frac{\partial^2 C}{\partial S^2} - h\frac{\partial C}{\partial t},$$

and a similar expression for a down move, except $-\sigma\sqrt{h}$ replaces $\sigma\sqrt{h}$. We can now replace the exponential functions and r^h with their Taylor series expansions. Of course, we could have done that along with the first step, and the separation is only to make the exposition clear. Here we would have, for example,

$$e^{\sigma\sqrt{h}} = 1 + \sigma\sqrt{h} + \tfrac{1}{2}\sigma^2 h + \tfrac{1}{6}\sigma^3(h)^{3/2} + \cdots.$$

By substituting these into the equation, collecting terms, and retaining only terms of order h, we obtain

$$\tfrac{1}{2}\sigma^2 hS^2\frac{\partial^2 C}{\partial S^2} + (\log r)hS\frac{\partial C}{\partial S} - h\frac{\partial C}{\partial t} - (\log r)hC = 0.$$

This form perhaps makes it easier to see why we did not bother with higher-order terms. If we had, to the above four terms we would have added R = [terms in $(h)^{3/2}$, $(h)^2$, ...]. If we then divide by h, we get

$$\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + (\log r)S\frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - (\log r)C + \frac{R}{h} = 0.$$

Now R/h goes to zero when h goes to zero, but the other terms do not, so we are left with the Black-Scholes equation.

Option Risk and Expected Rate of Return. We have just seen that the Black-Scholes formula is a limiting special case of the binomial formula we developed in Section 5-3. We showed in Section 5-5 how the *one-period* risk and expected return of an option were related to those of the stock for the binomial model. Since the results derived there did not depend on the length of the period, they must be valid for the Black-Scholes model as well. By pursuing arguments similar to those used earlier in this section, it can be shown that as the length of a period h becomes very small, the expected rate of return on the stock over the period approaches $(\mu + \frac{1}{2}\sigma^2)h$, and the variance of the rate of return over the period approaches $\sigma^2 h$. Naturally, the shorter the period, the smaller the risk and expected rate of return. In the same way, we could find the corresponding measures for a call. The results of Section 5-5 then tell us that these measures will be related to those of the stock in the following way:

> Expected rate of return of a call − riskless interest rate
> = Ω(expected rate of return of the stock − riskless interest rate)
>
> Volatility of a call = Ω(volatility of the stock)
>
> Beta of a call = Ω(beta of the stock)

where $\Omega = (S/C)\Delta$. To complete the analysis, we need to find the limiting value of Δ. By applying the Taylor series expansions discussed in the last subsection, we find that as $n \to \infty$ and $h \to 0$,

$$\Delta = \frac{C_u - C_d}{(u - d)S} \to \frac{\partial C}{\partial S},$$

where $\partial C/\partial S$ is the partial derivative of C with respect to S. Using the Black-Scholes formula, it can be shown that $\partial C/\partial S = N(x)$. In other words, a change in the stock price by the very small amount g, other things equal, causes the call value to change by $(\partial C/\partial S)g = N(x)g$. In summary, for the limiting Black-Scholes case, the *delta* and *elasticity* of a call are, respectively,

$$\Delta = N(x)$$

$$\Omega = \frac{SN(x)}{C}.$$

Note once again that the expected rate of return of a call *does* depend on the expected rate of return of the stock, even though the value of a call *does not*.

Two other concepts will prove useful later. The delta of a call clearly depends on the level of the stock price. We will denote the sensitivity of delta to changes in the stock price, as measured by the partial derivative, as the *gamma* of a call:

$$\Gamma \equiv \frac{\partial \Delta}{\partial S}.$$

Similarly, we will refer to the sensitivity of the value of a call to the passage of time as the call's *theta*. Since time to expiration t decreases as time passes, theta will be the negative of the partial derivative of C with respect to t:

$$\Theta \equiv -\frac{\partial C}{\partial t}.$$

Black-Scholes Put Valuation. The Black-Scholes formula for valuing European puts can be derived in a similar manner to the call formula. However, we can shortcut this procedure by combining this latter formula with the European put-call parity relationship for payout-protected options.

From Chapters 2 and 4, we recall that

$$P = C - S + Kr^{-t}.$$

This holds under the very general condition that no profitable riskless arbitrage opportunities exist. Since this assumption is consistent with the assumptions underlying the formula, we know this parity relationship must hold here as well.

Therefore, substituting for the Black-Scholes value of C, we have

$$P = -S[1 - N(x)] + Kr^{-t}[1 - N(x - \sigma\sqrt{t})].$$

Using the symmetry property of the standard normal distribution, we find the Black-Scholes put formula:

$$P = Kr^{-t}N(y + \sigma\sqrt{t}) - SN(y)$$

where

$$y \equiv \frac{\log(Kr^{-t}/S)}{\sigma\sqrt{t}} - \tfrac{1}{2}\sigma\sqrt{t}$$

For a put, $\Delta \equiv \partial P/\partial S = -N(y)$ and $\Omega = S\Delta/P$. Consequently, $\Delta \leq 0$ and $\Omega \leq 0$, and the volatility of the put, which must be positive, equals the negative of Ω times the volatility of the stock. Except for these changes, we have the same risk and return relationships that hold for calls. Since nothing in our discussion of the continuous-trading valuation equation specifically concerned a call, Equation (8) will also hold for a put.

Some Minor Generalizations. Thus far we have assumed the interest rate is known and constant over time. If, instead, the interest rate were predictably certain but different for different periods, then we would need to associate a different interest rate $r_k - 1$ with each period k. The same formula could then be derived, except that the discount factor r^{-t} in the formula is replaced by $1/(r_1 r_2 r_3 \cdots r_t)$, which in the continuous limit becomes $\exp(-\int_0^t \log r(v)\, dv)$.[25] Since we could have written r^{-t} as $\exp[-(\log r)t]$, this simply says that the constant interest rate $\log r$ is replaced by the average interest rate which will prevail over the remaining life of the option $\int_0^t \log r(v)\, dv/t$.

Likewise, the volatility could vary predictably with time. This implies the up and down movements u and d will depend on the date. In the limiting case, σ will depend on the date, and the variance of $\log(S^*/S)$ will be $\int_0^t \sigma^2(v)\, dv$ rather than $\sigma^2 t$. The Black-Scholes formula remains valid when $\sigma^2 t$ is replaced by this integral. Once again, this is a very sensible result. It says that the constant volatility σ is replaced by the average volatility which will prevail over the remaining life of the option, $[\int_0^t \sigma^2(v)\, dv/t]^{1/2}$.

Consequently, there is no difficulty including interest rates and volatility that change over time in a predictable way. However, if future interest rates or volatility cannot be predicted with certainty, then our option pricing approach requires more serious modification. We will return to this possibility in Chapter 7.

5-7. AN ALTERNATIVE DERIVATION

This section contains a brief description of an alternative approach to deriving the Black-Scholes formula.[26] It shows how the Black-Scholes formula can be derived directly from the more traditional discrete-time, general equilibrium models used in the theory of finance.

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[25] The notation $\exp(z)$ means e raised to the power z.

[26] This section is not necessary for understanding subsequent chapters. Also, it presumes some familiarity with the capital asset pricing model. For these reasons, the reader may wish to skip directly to Section 5-8.

The now traditional form of the capital asset pricing model says that, under certain circumstances, the current price X of *any* security is determined by

$$X = \frac{E(X^*) - \lambda \operatorname{Cov}(X^*, r_M)}{r},$$

where X^* is its (uncertain) price at the end of the period, λ is a positive constant, $r_M - 1$ is the rate of return on the market portfolio, and $r - 1$ is the rate of interest over the period. E and Cov denote expectation and covariance, respectively.

Since this holds for any security, for an underlying stock and its call option,

$$S = \frac{E(S^*) - \lambda \operatorname{Cov}(S^*, r_M)}{r} \quad \text{and} \quad C = \frac{E(C^*) - \lambda \operatorname{Cov}(C^*, r_M)}{r},$$

where

$$C^* = \max[0, S^* - K].$$

These equations link the option and stock together. Remembering that S^* and r_M are jointly normally distributed, we can hope to use them to derive an option pricing formula relating C to S.

Although this can be done, the theory has two critical disadvantages. First, the theory assumes the joint distribution of *all* available securities is multivariate normal. However, by the contractual provisions of a call (that is, $C^* = \max[0, S^* - K]$, C^* cannot be normally distributed even if S^* is normally distributed. Moreover, if S^* is normally distributed, among other unfortunate implications, it cannot also have limited liability. Second, the option pricing problem is inherently multiperiod, where the purchaser of an option has many opportunities to sell or exercise it before it expires. The classical capital asset pricing model is essentially a single-period theory and does not conveniently accommodate opportunities for portfolio revision before a terminal date.[27]

To create a satisfactory theory, yet one that does not require continuous trading or binomial outcomes, one can replace the normality restriction on security returns with a logarithmic utility assumption on investor preferences. If this is done,[28] the following multiperiod formula

[27] Attempts to place the model in a useful multiperiod context require the further assumption that λ is an intertemporal constant.

[28] See Mark Rubinstein, "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics*, 7 (Autumn 1976), 407–425.

replaces the usual capital asset pricing model:

$$X = \sum_{k=1}^{\infty} \frac{E(D_k) - \lambda_k \operatorname{Cov}(D_k, -r_M^{-k})}{r^k},$$

where

$$\lambda_k = \sqrt{(1 + \lambda^2)^k - 1} \qquad \text{and} \qquad \lambda \equiv [E(r_M^{-1})]^{-1}.$$

D_k represents the (uncertain) cash distribution received on date k from the security.

For non-dividend-paying stock over time t, this formula simplifies to

$$S = \frac{E(S^*) - \lambda_t \operatorname{Cov}(S^*, -r_M^{-t})}{r^t},$$

and for a call with time to expiration t,

$$C = \frac{E(C^*) - \lambda_t \operatorname{Cov}(C^*, -r_M^{-t})}{r^t},$$

where

$$C^* = \max[0, S^* - K].$$

This is quite similar to the three relationships derived from the capital asset pricing model, except we have not imposed the disagreeable stochastic restrictions on S^* and C^*, and have accounted for the multiperiod nature of options markets.[29]

However, without some stochastic assumption governing S^*, we cannot completely solve the problem of finding a formula for C in terms of S. We need to know something about the probability that it will pay to exercise the call. Since we are free to adopt whatever stochastic restriction we wish, we will choose a reasonable one: S^* and r_M are *jointly lognormally* distributed. Although this is a stronger stochastic assumption than that made in our original derivation in Section 5-6, since it implies S^*/S will itself be lognormally distributed, it is consistent with that derivation.

Since the proof is tedious, it is not repeated here. Suffice it to say that the above three equations, together with the joint lognormality assumption, yield a result *identical with the Black-Scholes formula.* At first thought, this is quite surprising, since neither continuous trading nor binomial outcomes have been assumed. Rather, investors are only permitted to trade at discrete points in time and, at each point, the stock price—being lognormal—can have any one of an infinite number of values. Investors are thus unable to

[29] Moreover, unlike the capital asset pricing model, investor agreement about the joint probability distribution of security returns is also not required.

construct riskless hedges with the option and the stock, a capability that was crucial to previous proofs. Indeed, it was precisely to circumvent a riskless hedging argument that this alternative model was created.

A simple explanation of this anomaly can be found in portfolio theory: Logarithmic utility is the only utility function for which portfolios are chosen independently of opportunities to revise them in the future. Therefore, the relative pricing relationship between a European option and its underlying stock will be independent of the number of times portfolio revision can occur before the expiration date of an option. In particular, the same relationship will hold even if the investor faces continuous revision opportunities. But we have already shown that assumption leads directly to the Black-Scholes formula.

In summary, the discrete-time logarithmic utility model, by reaching the same option pricing conclusion as a hedging model, indicates the robustness of the Black-Scholes formula to its assumption of continuous trading. To underscore the significance of this result, consider an investor who, for some reason, cannot implement a dynamic riskless hedging strategy similar to that described in Section 5-4. Nonetheless, he may very well *value* an option according to the Black-Scholes formula. Only at the Black-Scholes price will a static position in the option provide fair compensation in terms of expected return for the risk borne.

5-8. HOW CHANGES IN THE VARIABLES AFFECT BLACK-SCHOLES OPTION VALUES

Extreme Values. One way to understand the formula is to examine what happens to the call value C as the variables S, K, t, σ, and r, on which it depends, change in value. To keep the effects clear, we will choose one of these five variables and change its value, while holding the other four fixed. First, we examine extreme changes:

1. Stock price:
   as $S \to 0$, then $C \to 0$
   as $S \to \infty$, then $C \to \infty$
2. Striking price:
   as $K \to 0$, then $C \to S$
   as $K \to \infty$, then $C \to 0$
3. Time to expiration:
   given $S < K$: as $t \to 0$, then $C \to 0$
   given $S > K$: as $t \to 0$, then $C \to S - K$
   as $t \to \infty$, then $C \to S$

4. Volatility:
   given $S < Kr^{-t}$: as $\sigma \to 0$, then $C \to 0$
   given $S > Kr^{-t}$: as $\sigma \to 0$, then $C \to S - Kr^{-t}$
   as $\sigma \to \infty$, then $C \to S$
5. Interest rate:
   as $r \to \infty$, then $C \to S$

The reader should try to prove each of these assertions. To take the most difficult case, as $\sigma \to \infty$, then $x = [\log(S/Kr^{-t})/\sigma\sqrt{t}] + \frac{1}{2}\sigma\sqrt{t} \to +\infty$ and $x - \sigma\sqrt{t} \to -\infty$. Since

$$N(x) \to N(+\infty) = 1 \qquad \text{and} \qquad N(x - \sigma\sqrt{t}) \to N(-\infty) = 0,$$

then $C \to S$.

All these implications of the formula are fully consistent with the general arbitrage relationships developed in Chapter 4. Of course, if they were not, our exact formula would be in error.

Tabular Representation. Table 5-2 gives formula-generated values for nine calls in a typical option class with current stock price of \$40. The class includes out-of-, at-, and in-the-money series ranging over near, middle, and far maturities. Values for the nine series cover low, middle, and high volatilities, and low, middle, and high interest rates. As we would expect from Chapter 4, call values increase with lower striking price, longer time to expiration and higher volatility.

Table 5-2
REPRESENTATIVE BLACK-SCHOLES CALL VALUES

		$S = 40$								
		$r = 1.03$			$r = 1.05$			$r = 1.07$		
σ	K	*Expiration Month*								
		JAN	*APR*	*JUL*	*JAN*	*APR*	*JUL*	*JAN*	*APR*	*JUL*
.2	35	5.09	5.56	6.08	5.15	5.76	6.40	5.20	5.95	6.71
	40	.97	2.04	2.77	1.00	2.17	3.00	1.04	2.30	3.24
	45	.02	.46	.98	.02	.51	1.10	.02	.56	1.23
.3	35	5.17	6.08	6.90	5.22	6.25	7.17	5.27	6.42	7.44
	40	1.43	2.95	3.97	1.46	3.07	4.19	1.49	3.20	4.40
	45	.16	1.19	2.09	.16	1.25	2.24	.17	1.33	2.39
.4	35	5.34	6.74	7.85	5.39	6.89	8.09	5.44	7.05	8.34
	40	1.89	3.86	5.16	1.92	3.98	5.37	1.95	4.10	5.58
	45	.41	2.02	3.27	.42	2.10	3.43	.43	2.18	3.59

NOTE: The January options have one month to expiration; the Aprils, four months; and the Julys, seven months. Both r and σ are expressed in annual terms.

An unresolved issue in that chapter was the effect of the interest rate. Although it lacks the status of a general arbitrage relationship, within the context of the Black-Scholes formula, call values appear to increase with higher interest rates. Indeed, this can be directly verified analytically. This result is anticipated, since r enters the formula inversely to K. However, the same percentage change in the interest rate $r - 1$ has a much smaller effect on call values than a percentage change in the striking price, time to expiration, or volatility. For example, for the at-the-money, middle-maturity calls at 5% interest rate, doubling the volatility from .2 to .4 increases the option value from \$2.17 to \$3.98. In contrast, for the at-the-money, middle-maturity calls at .3 volatility, more than doubling the interest rate from 3% to 7% only raises the option value from \$2.95 to \$3.20.

Graphical Representation. Figures 5-4 through 5-11 show how the formula values for typical out-of-, at-, and in-the-money calls and puts change as the current stock price, time to expiration, volatility and interest rate change gradually over wide ranges. Figures 5-4 and 5-8 are the option-stock price diagrams for a call and a put, respectively. Just as we would expect, the properties shown in the graphs correspond exactly to those given in Chapter 4. In all the graphs, time is measured in years. Thus σ and r are expressed in annualized terms.

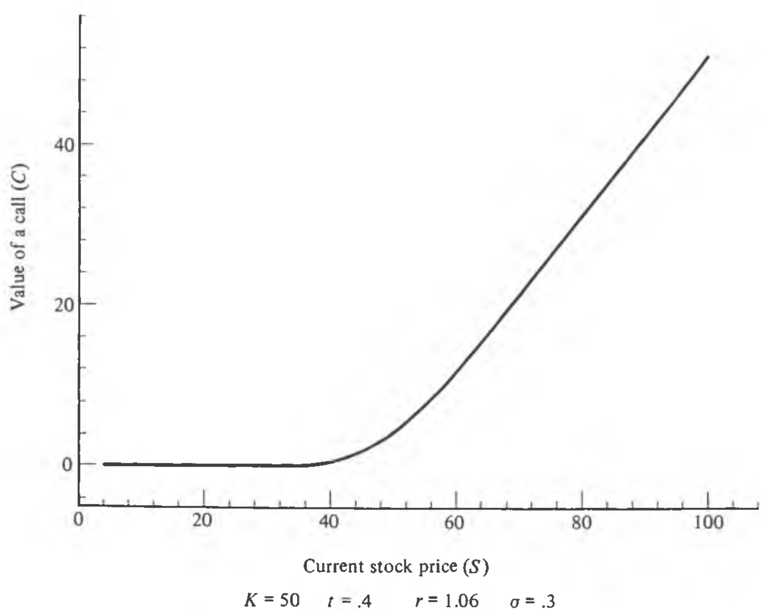


Figure 5-4 The Value of a Call as a Function of the Current Stock Price

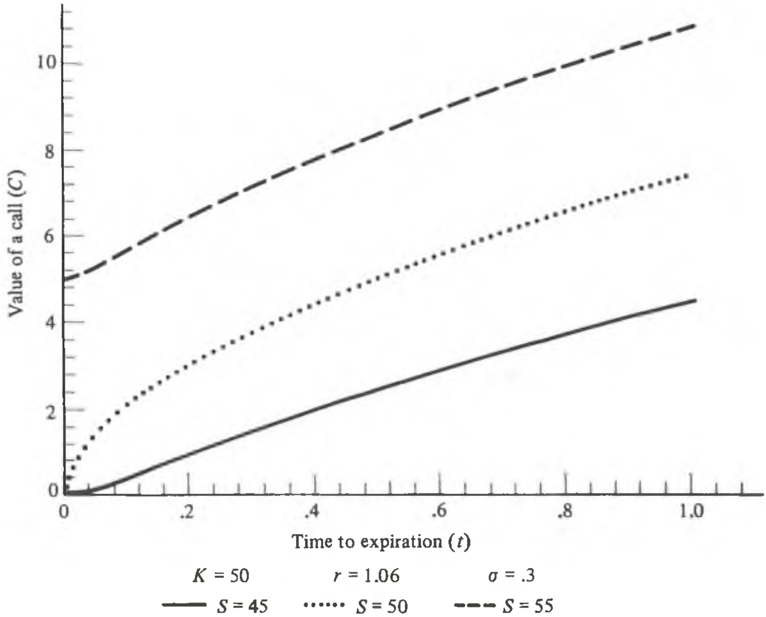


Figure 5-5 The Value of a Call as a Function of the Time to Expiration

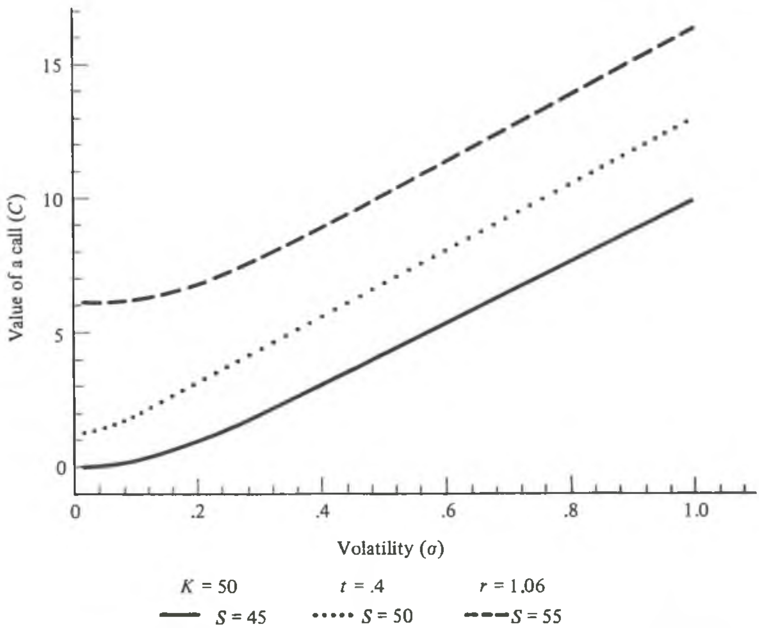


Figure 5-6 The Value of a Call as a Function of the Volatility

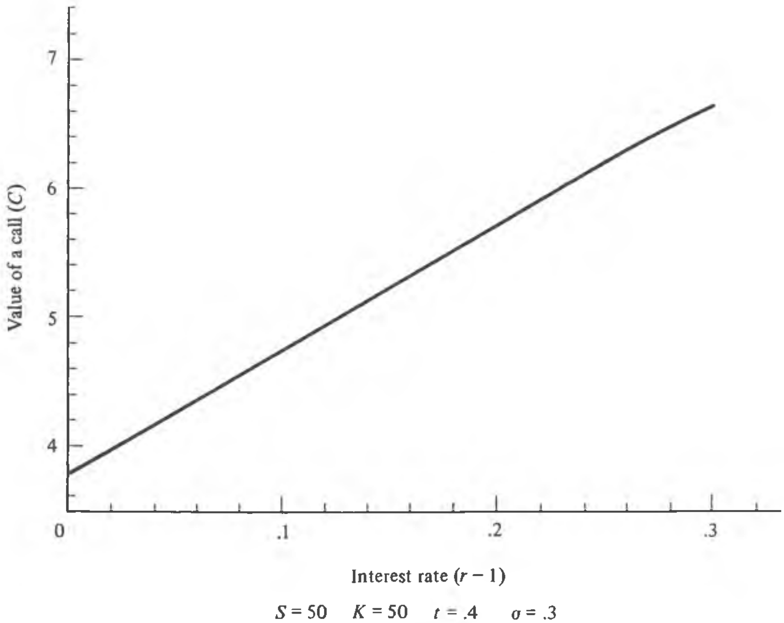


Figure 5-7 The Value of a Call as a Function of the Interest Rate

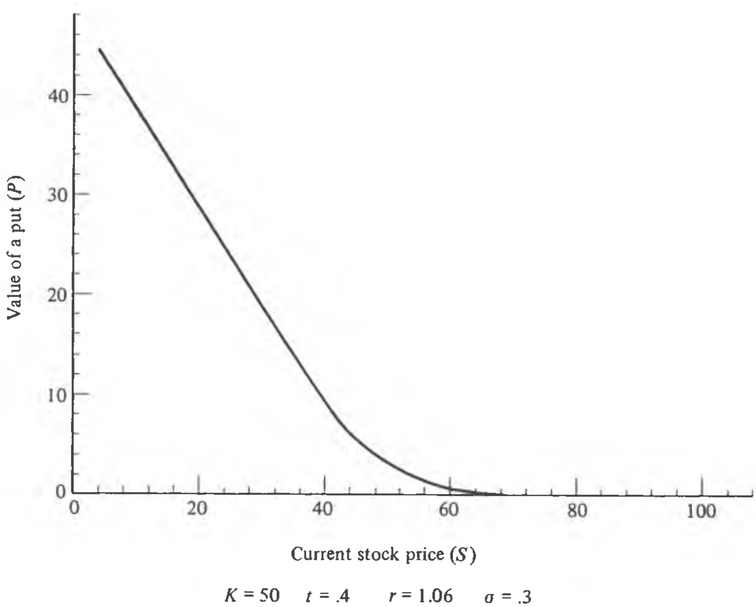


Figure 5-8 The Value of a Put as a Function of the Current Stock Price

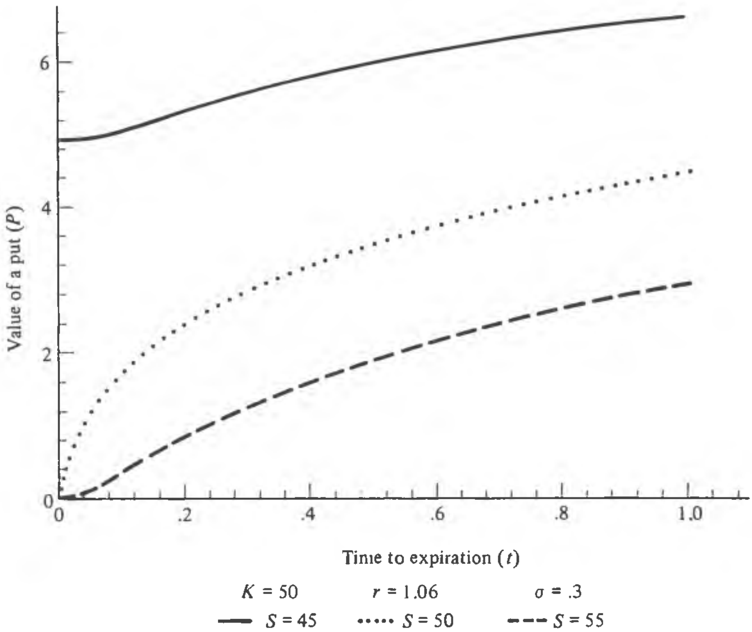


Figure 5-9 The Value of a Put as a Function of the Time to Expiration

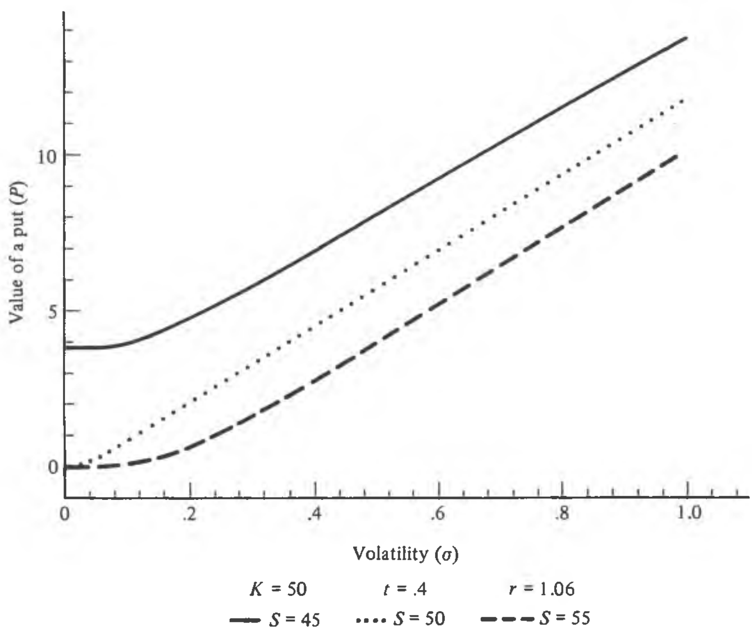


Figure 5-10 The Value of a Put as a Function of the Volatility

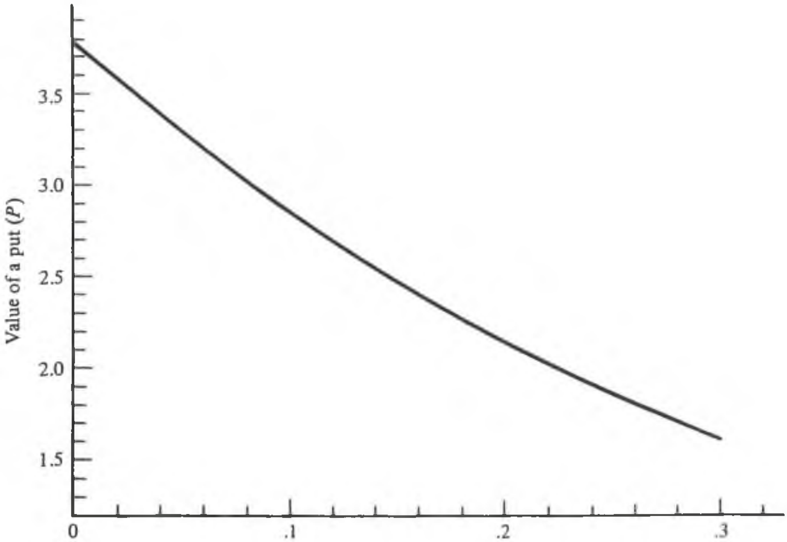


Figure 5-11 The Value of a Put as a Function of the Interest Rate

The *sensitivity* of the formula call value to small changes in each of the five determining variables gives an alternative mathematical representation:[30]

1. Stock price: $\partial C/\partial S = N(x) > 0$
2. Striking price: $\partial C/\partial K = -r^{-t}N(x - \sigma\sqrt{t}) < 0$
3. Time to expiration:
   $\partial C/\partial t = (S\sigma/2\sqrt{t})N'(x) + Kr^{-t}(\log r)N(x - \sigma\sqrt{t}) > 0$
4. Volatility: $\partial C/\partial \sigma = S\sqrt{t}N'(x) > 0$
5. Interest rate: $\partial C/\partial r = tKr^{-(t+1)}N(x - \sigma\sqrt{t}) > 0$

[30] Readers familiar with calculus who would like to verify these derivatives should use the fact

$$\frac{\partial N(z)}{\partial v} = N'(z)\frac{\partial z}{\partial v},$$

where $N'(z)$ is the standard normal density function evaluated at z. That is, $N'(z) = (1/\sqrt{2\pi})e^{-z^2/2}$.

In other words, for example, a change in the volatility of the stock by the small amount g causes the call value to change by $(\partial C/\partial \sigma)g = S\sqrt{t}\ N'(x)g$. The sensitivity $\partial C/\partial S$ is, of course, the option delta, and the sensitivity $-\partial C/\partial t$ is the option theta.

Figures 5-12 through 5-19 show how these sensitivities for a call vary with the current stock price and time to expiration.[31] Figure 5-12 shows that the call delta will be near zero for deep-out-of-the-money calls and

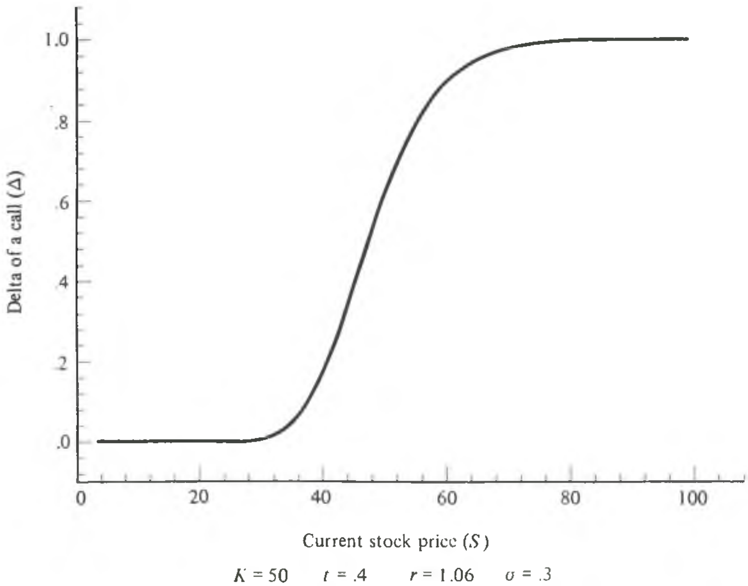


Figure 5-12 The Delta of a Call as a Function of the Current Stock Price

near one for deep-in-the-money calls. The delta only changes quickly with the current stock price for calls near-the-money. Figure 5-13 indicates that the call delta falls as the expiration date approaches for out-of-the-money calls but tends to rise for in-the-money calls. From Figure 5-14 we learn that changes in time to expiration have the greatest dollar effect on calls near-the-money. In Chapter 4, from pure arbitrage considerations, we were unable to say if, for three otherwise identical calls, the middle maturity call should sell for more than half the sum of the near- and far-maturity calls.

[31] The reader familiar with calculus should recognize the slopes of these figures as representing second and cross-partial derivatives.

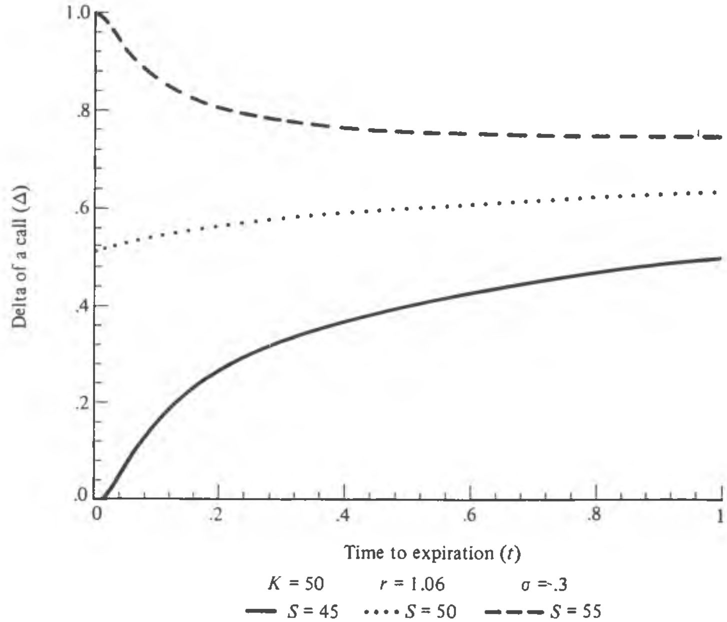


Figure 5-13 The Delta of a Call as a Function of the Time to Expiration

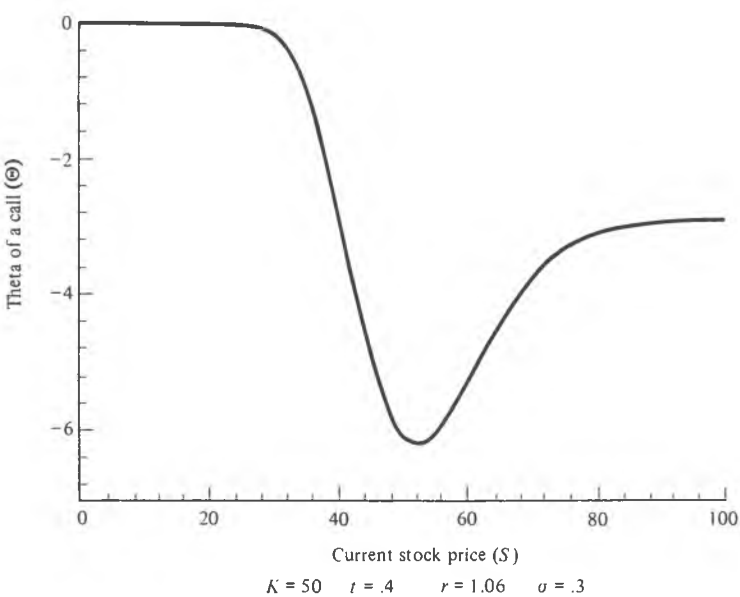


Figure 5-14 The Theta of a Call as a Function of the Current Stock Price

Figure 5-15 gives the answer in the context of the Black-Scholes formula: This relationship will hold except for out-of- or in-the-money calls when the near-maturity call is very close to expiration. Figures 5-16 and 5-17 show that calls with values most sensitive to volatility are near-the-money and of long maturity, while Figures 5-18 and 5-19 show that the values of deep-in-the-money long-maturity calls are the most sensitive to the interest rate.

For the Black-Scholes model, the gamma and elasticity of a call are

$$\Gamma \equiv \frac{\partial \Delta}{\partial S} = \frac{1}{S\sigma\sqrt{t}} N'(x),$$
$$\Omega \equiv \frac{S\Delta}{C} = \frac{SN(x)}{C}.$$

Figures 5-20 through 5-23 show how these values depend on the current stock price and time to expiration.

Figure 5-20 indicates that the delta of a call will be most sensitive to changes in the stock price when the call is slightly out-of-the-money. Further calculations with the formula confirm this and show that gamma

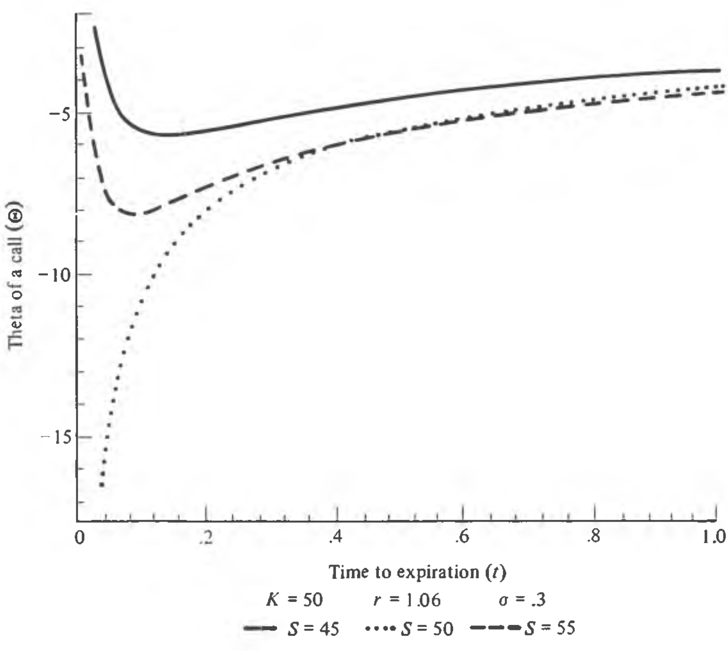


Figure 5-15 The Theta of a Call as a Function of the Time to Expiration

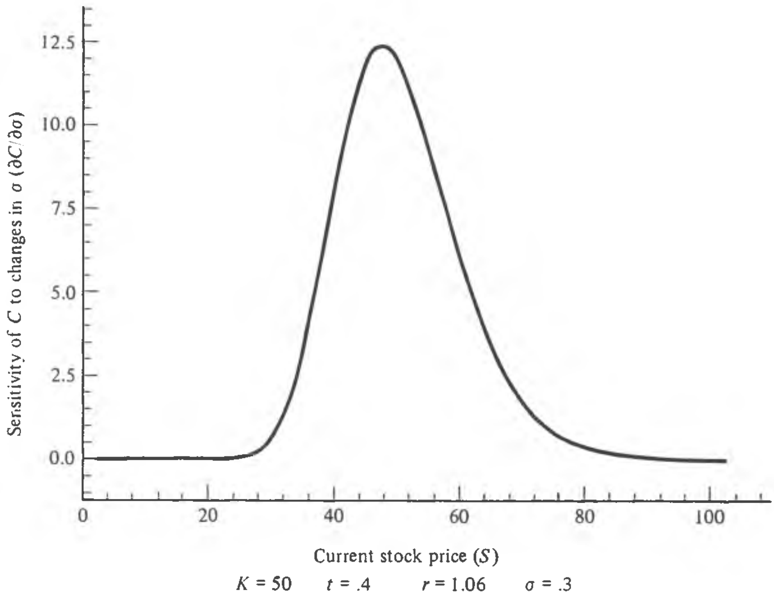


Figure 5-16 The Sensitivity of C to Changes in σ as a Function of the Current Stock Price

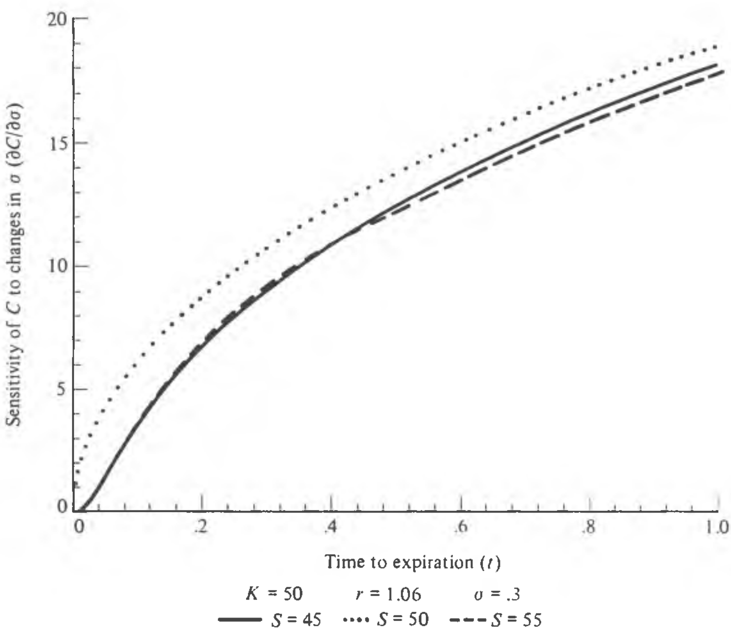


Figure 5-17 The Sensitivity of C to Changes in σ as a Function of the Time to Expiration

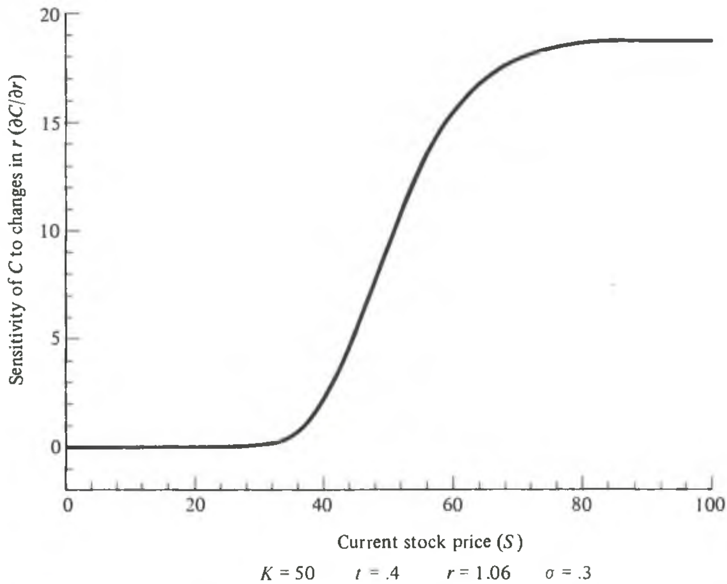


Figure 5-18 The Sensitivity of C to Changes in r as a Function of the Current Stock Price

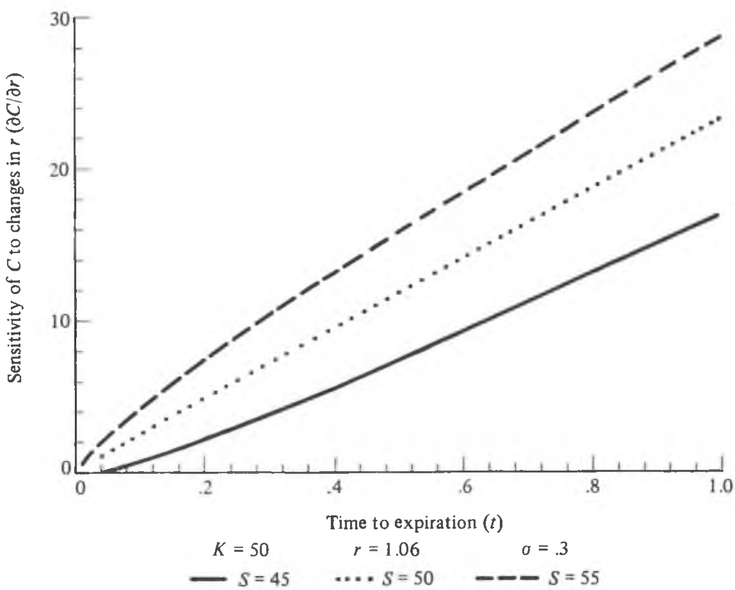


Figure 5-19 The Sensitivity of C to Changes in r as a Function of the Time to Expiration

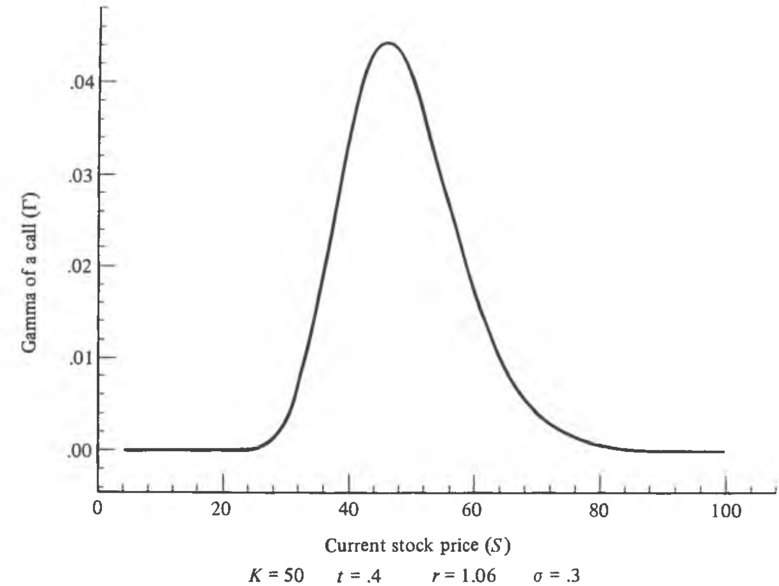


Figure 5-20 The Gamma of a Call as a Function of the Current Stock Price

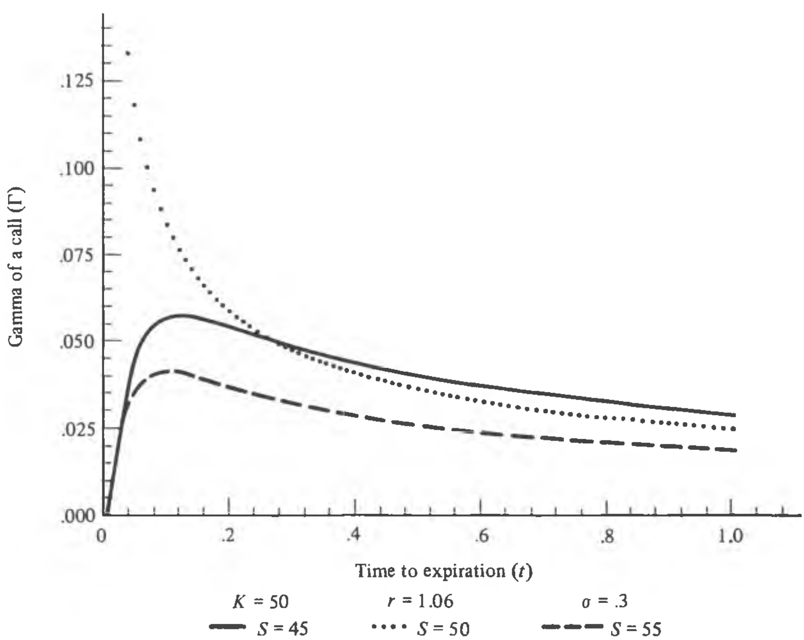


Figure 5-21 The Gamma of a Call as a Function of the Time to Expiration

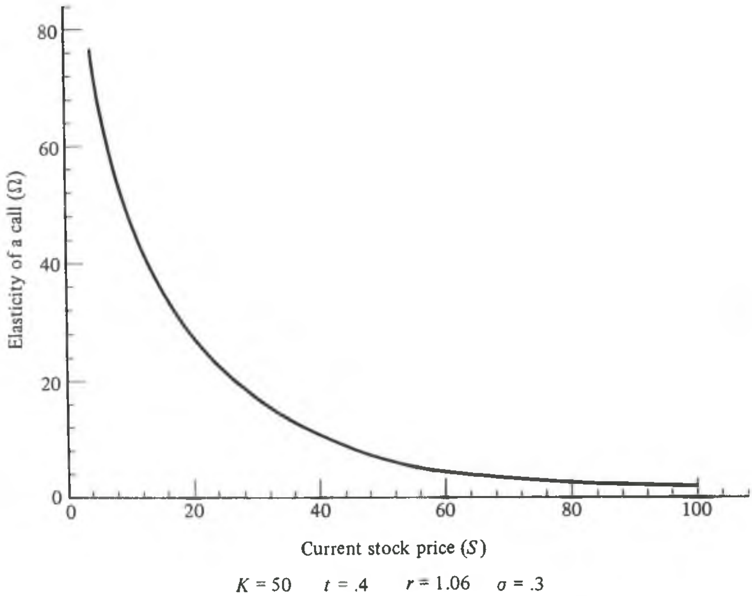


Figure 5-22 The Elasticity of a Call as a Function of the Current Stock Price

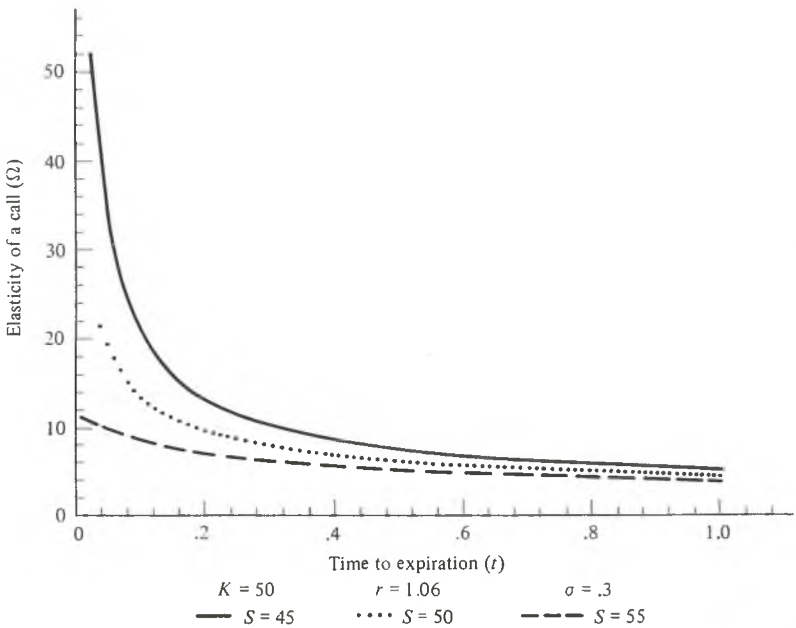


Figure 5-23 The Elasticity of a Call as a Function of the Time to Expiration

reaches its largest value when $S = Kr^{-t}[\exp(-\frac{3}{2}\sigma^2 t)]$. Figure 5-21 shows that, for a given stock price, gamma will go to zero as the expiration date approaches if the call is in-the-money or out-of-the-money, but will become very large if it is exactly at-the-money. For an at-the-money call with a short lifetime, a very small change in the stock price can lead to a significant change in the composition of the equivalent portfolio.

Remember that the elasticity of an option shows the percentage change in its value that will accompany a small percentage change in the stock price. Figure 5-22 shows that the elasticity increases as the stock price decreases. Taking an extreme case, according to Figure 5-22, a call 30 points out-of-the-money (that is, $S = \$20$) changes about 28% in value for a 1% change in the stock price. It is certainly no overstatement to say that deep-out-of-the-money calls are highly levered securities. Figure 5-23 shows that the elasticity also increases as the time to expiration decreases. Other things equal, a call will be more sensitive to stock price movements *in percentage terms* the shorter the time remaining until expiration.

Figures 5-24 through 5-35 show comparable relationships for puts. The sensitivities for puts are

1. Stock price: $\partial P/\partial S = (\partial C/\partial S) - 1 < 0$
2. Striking price: $\partial P/\partial K = (\partial C/\partial K) + r^{-t} > 0$
3. Time to expiration:
   $\partial P/\partial t = (\partial C/\partial t) - (\log r)Kr^{-t} \gtrless 0$
4. Volatility: $\partial P/\partial \sigma = \partial C/\partial \sigma > 0$
5. Interest rate: $\partial P/\partial r = (\partial C/\partial r) - tKr^{-(t+1)} < 0$

Again $\partial P/\partial S$ is the put delta and $-\partial P/\partial t$ the put theta. As expected, the sign of the theta for European puts is ambiguous. Also, note that the interest rate affects put and call values in opposite directions.

For puts,

$$\Gamma \equiv \frac{\partial \Delta}{\partial S} = \frac{1}{S\sigma\sqrt{t}} N'(x)$$

and

$$\Omega \equiv (S/P)\Delta = \frac{S}{P}[N(x) - 1].$$

Observe that the gammas of puts and calls with identical terms are equal. This is confirmed by comparing Figures 5-20 and 5-21 with Figures 5-32 and 5-33.

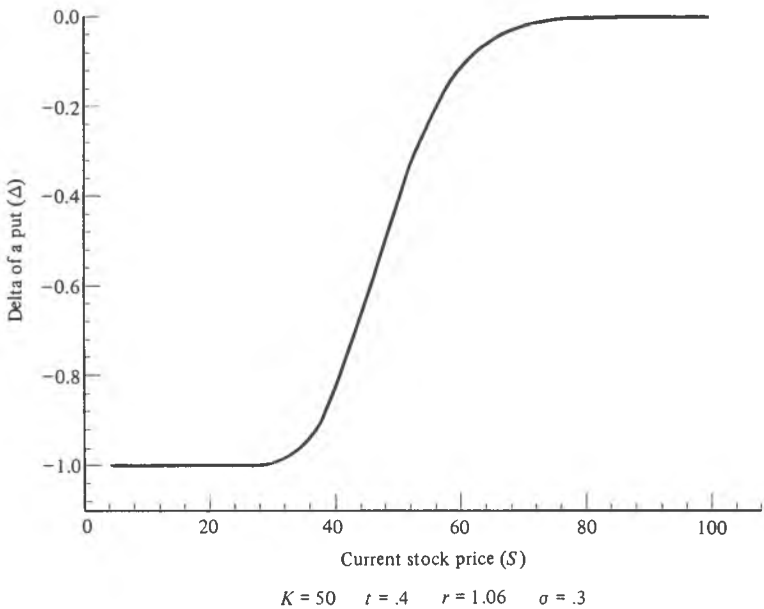


Figure 5-24 The Delta of a Put as a Function of the Current Stock Price

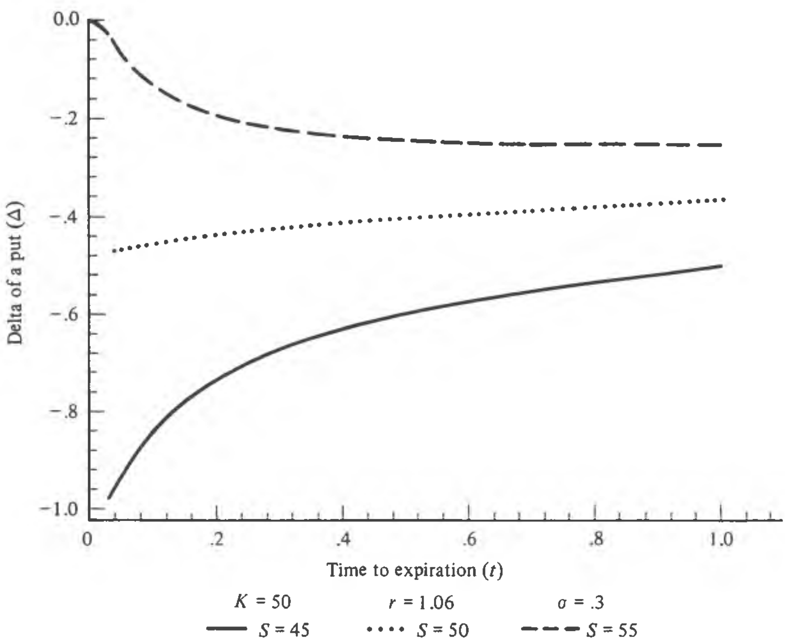
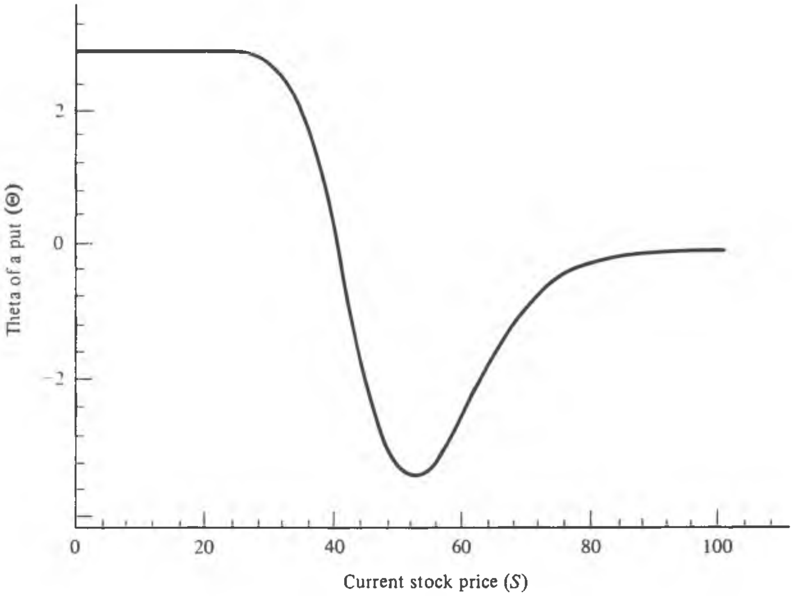
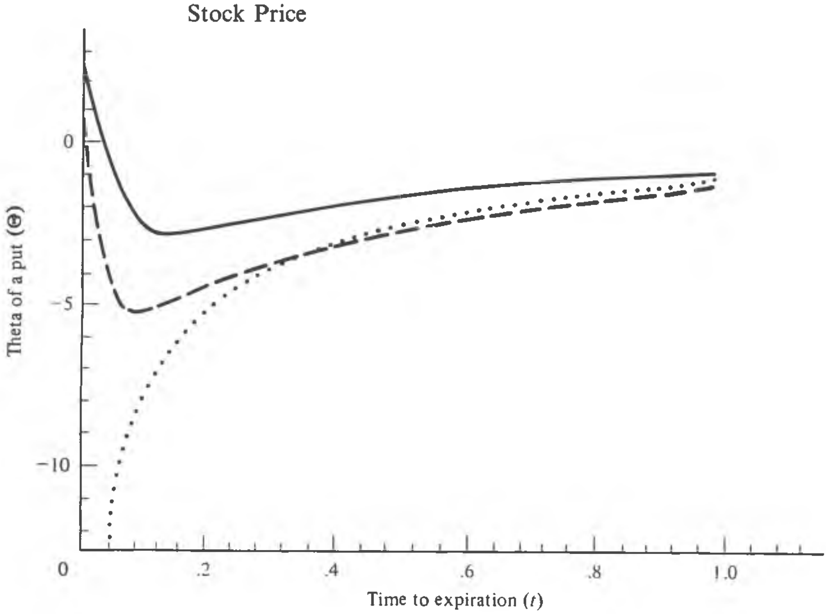


Figure 5-25 The Delta of a Put as a Function of the Time to Expiration



$K = 50 \quad t = .4 \quad r = 1.06 \quad \sigma = .3$

Figure 5-26 The Theta of a Put as a Function of the Current Stock Price



$K = 50 \quad r = 1.06 \quad \sigma = .3$

— $S = 45$ ···· $S = 50$ – – – $S = 35$

Figure 5-27 The Theta of a Put as a Function of the Time to Expiration

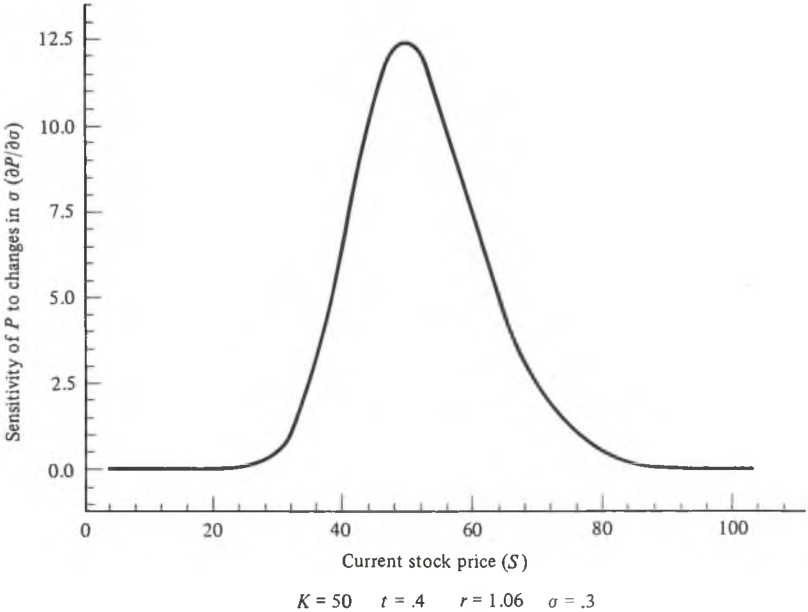


Figure 5-28 The Sensitivity of P to Changes in σ as a Function of the Current Stock Price

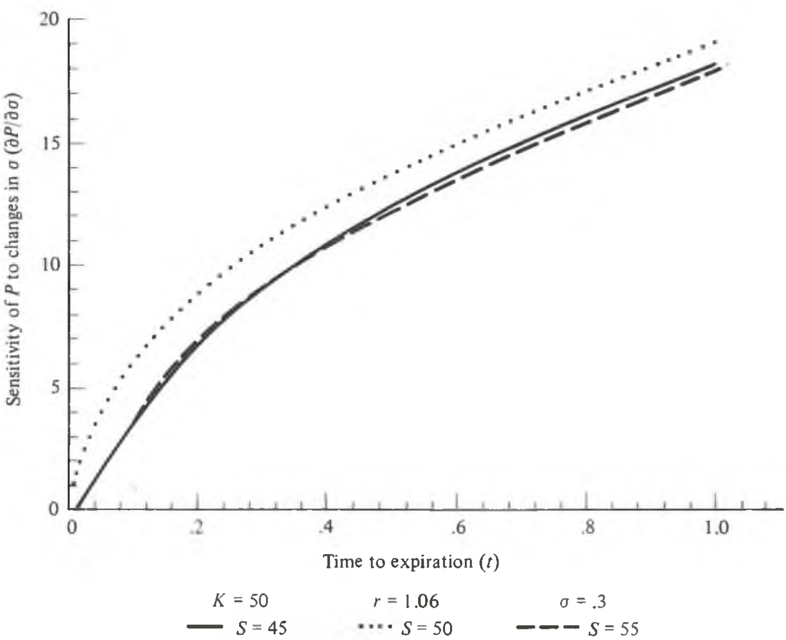


Figure 5-29 The Sensitivity of P to Changes in σ as a Function of the Time to Expiration

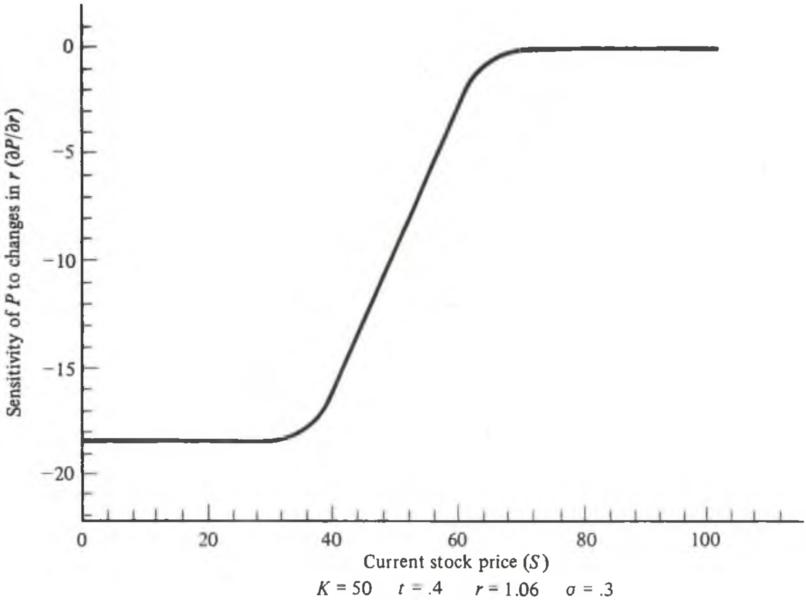


Figure 5-30 The Sensitivity of P to Changes in r as a Function of the Current Stock Price

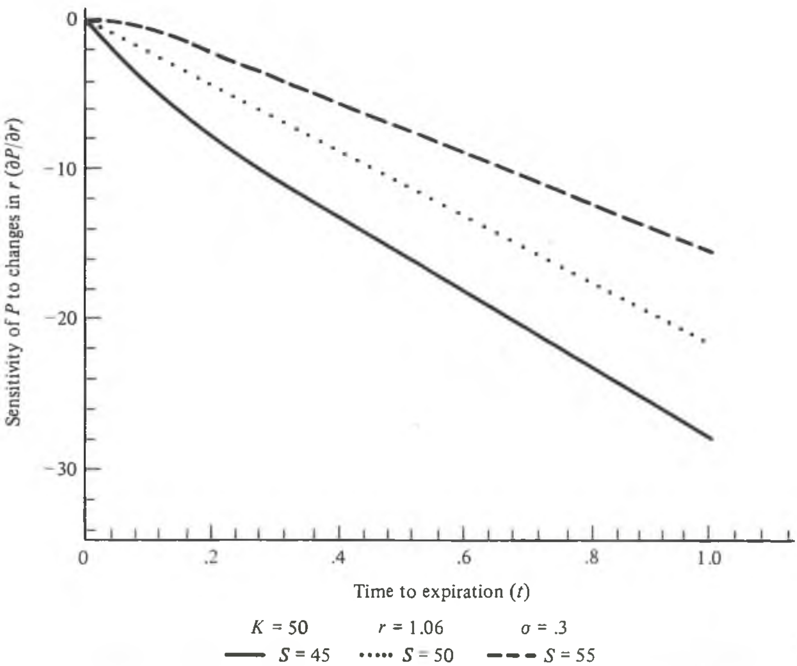


Figure 5-31 The Sensitivity of P to Changes in r as a Function of the Time to Expiration

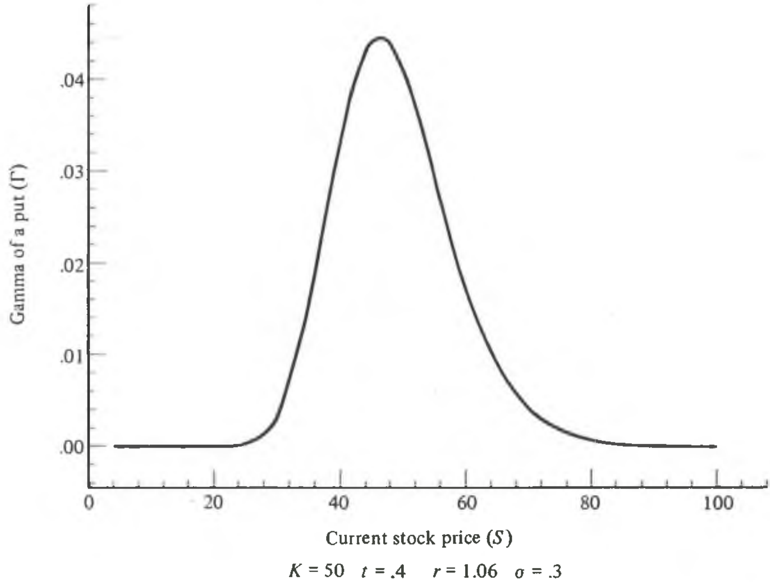


Figure 5-32 The Gamma of a Put as a Function of the Current Stock Price

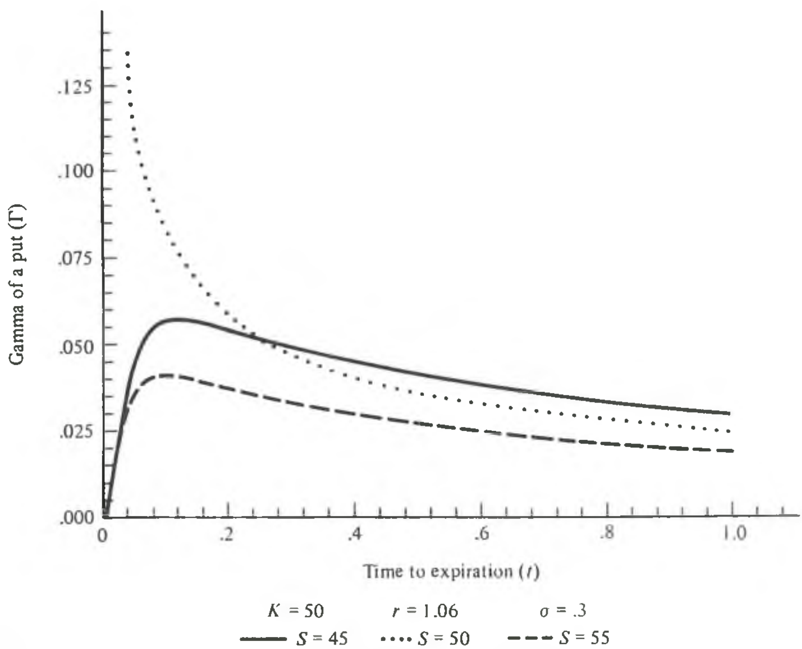


Figure 5-33 The Gamma of a Put as a Function of the Time to Expiration

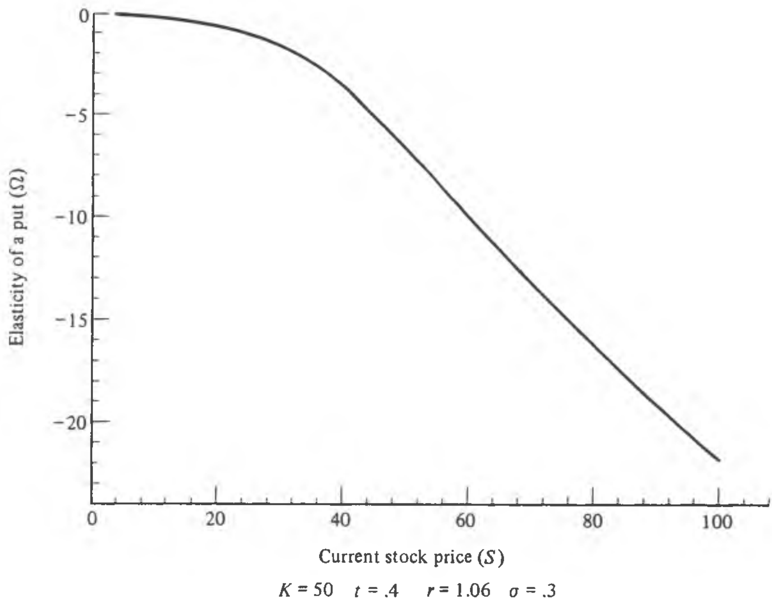


Figure 5-34 The Elasticity of a Put as a Function of the Current Stock Price

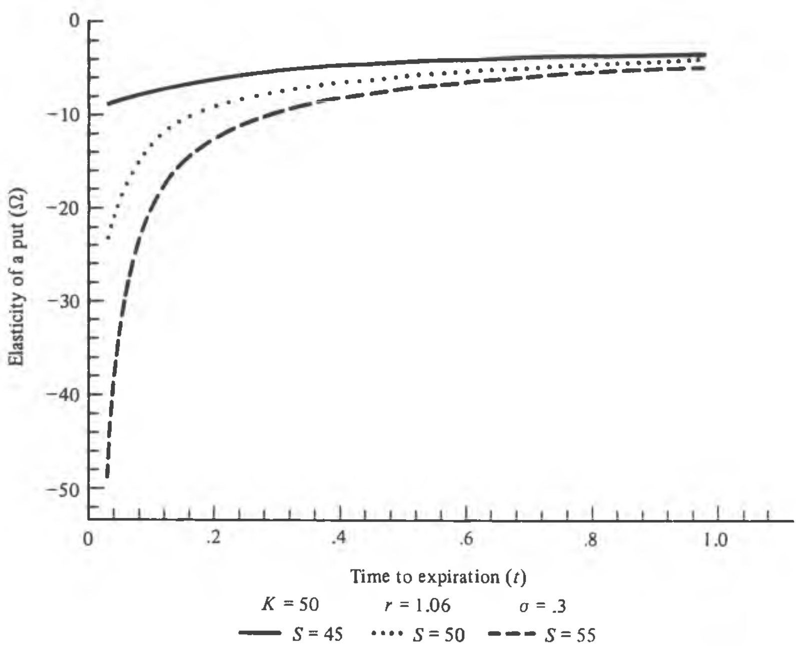


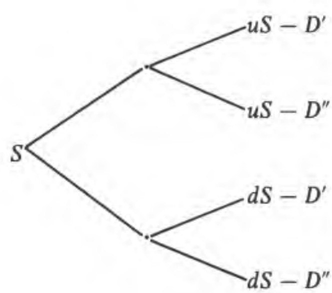
Figure 5-35 The Elasticity of a Put as a Function of the Time to Expiration

5-9. HOW TO INCLUDE DIVIDENDS AND THE POSSIBILITY OF EARLY EXERCISE

As we have seen, the value of an option with no payout protection will be affected if the stock pays cash dividends. Furthermore, it may then be optimal to exercise an American call before the expiration date. This possibility causes the value of an American call to exceed the value of an otherwise identical European call. Although cash dividends are not necessary for early exercise to be optimal for an American put, they will nevertheless affect the optimal exercise strategy.

To handle the effects of cash dividends with a riskless hedging argument, we must assume that the dividends that will be paid on any future date are a known function of, at most, the path followed by the stock price up to that date. This condition is actually not very restrictive. It allows sufficient flexibility to represent the dividend behavior of most firms. In particular, current and future dividends could be influenced in a quite general way by the sequence of dividends paid in the past. It requires us to know the dividend policy of the firm, but it certainly does not imply that we must know today exactly what future dividends will actually be, since they may depend on the currently unknown values of future stock prices.

It does, however, rule out random changes in the dividend level which are not perfectly correlated with the stock price. The critical feature was that, in every period, the ex-dividend stock price at the end of the period can take on only two possible values. Suppose, instead, that a dividend D on an ex-dividend date one period from now can either be D' or D'', where $D' < D''$, and that the size of the dividend is independent of the stock price. Then, the stock price could have four possible values one period hence:



For example, if, in the absence of dividends, the stock price would move up to uS, then the ex-dividend price will be $uS - D'$ or $uS - D''$. This uses the fact that to prevent arbitrage, the stock price must fall by the amount of the dividend. With *four* possible ex-dividend values of the stock price at the end of one period, we cannot set up a riskless arbitrage position using only *three* securities—the stock, a call, and a default-free security. If we try to choose a hedge ratio Δ that will equate the returns in two of the four outcomes, we will still be exposed to risk in the remaining two outcomes.

To make the following discussions more concrete, we will consider as an example a specific and particularly important dividend policy: The stock maintains a constant yield $\delta \equiv D/S$ on each ex-dividend date. Suppose there is one period remaining before expiration and the current stock price is S. If the end of the period is an ex-dividend date, then an individual who owned the stock during the period will receive at that time a dividend of either δuS if the stock price goes up to uS or δdS if the stock price goes down to dS. Hence, the stock price at the end of the period can take on only two possible values: $(1 - \delta)uS$ or $(1 - \delta)dS$.

$$S \begin{array}{l} \nearrow (1-\delta)uS \\ \searrow (1-\delta)dS \end{array}$$

In particular, if $\delta = .05$, then 5% of the value of the stock on an ex-dividend date would be paid out in dividends, leaving $1 - .05$ or 95% of the stock value remaining.

Although we will not pursue it further, a very similar setting can be used to illustrate more complicated policies that depend on past stock prices. For example, suppose that the dividend yield paid at the end of the current period will be 7% if the stock price moved upward in the previous period and 3% if it moved downward. The dividend policy will then depend on both the current stock price and the stock price last period.

Dividends and European Options. An *unprotected European option* written on a stock with a constant dividend yield δ is valued identically to one written on a stock paying no dividends, except that the current stock price *S is replaced with $S(1 - \delta)^{\bar{v}}$*, where $\bar{v}$ is the number of ex-dividend dates prior to the expiration of an option. To see this, consider any path the stock price may take over time. For instance, suppose, in the absence of dividends, the stock price at expiration $S^* = uudduuududS = u^6d^4S$, implying a down movement was followed by an up movement, which was followed by a down movement, etc. Suppose an ex-dividend date occurs every third period. Then,

$$S^* = u(1 - \delta)udd(1 - \delta)uuu(1 - \delta)dudS = u^6d^4(1 - \delta)^3S.$$

Now consider another path, $S^* = dudddudddduS = u^3d^7S$. Again, with an ex-dividend date every third period, this becomes:

$$S^* = d(1-\delta)udd(1-\delta)dud(1-\delta)dduS = u^3d^7(1-\delta)^3S.$$

Unlike an American option, the current value of a European option depends only on its possible values at expiration, which in turn depend only on the possible prices of its associated stock at expiration. In particular, given the same ending value, the path the stock price takes over time is irrelevant. As we can see, with a constant dividend yield, the stock price at expiration depends on the *number* $\bar{v}$ of ex-dividend dates, but it does not depend on the *timing* of the dividends. Therefore, *we will obtain identical values for unprotected European options whether we consider the actual path taken by the stock price or regard the dividends as all paid immediately.*

American Calls. The possible early exercise of unprotected American calls implies their current value depends on the path of the stock price through expiration and thereby on the timing of the ex-dividend dates. We know from Section 4-1 that the only time early exercise of calls should be considered is just prior to their ex-dividend dates. If the call holder exercises his call just prior to the ex-dividend date, he receives the dividend on the stock that he would otherwise forego. Exercise between ex-dividend dates sacrifices premium over parity unnecessarily.

One way to take early exercise into account is to calculate for each listed call several European values, one for early exercise just before each ex-dividend date prior to expiration, and one assuming exercise at expiration. Shortening the life of the call tends to decrease its value, but deducting fewer dividends tends to increase it. The estimated, or "pseudo-American," current call value is then simply the highest of these European values.

Unfortunately, this method of adjusting American calls for dividends understates their actual values. For example, if the highest European value is based on exercising just before the last ex-dividend date prior to expiration, then the method presumes that exercise policy will remain optimal. But, if the stock price experiences a sudden decline, pushing the call out-of-the-money, early exercise may no longer be advisable. The additional flexibility actually afforded by an American call to change our minds about exercise means that an American call will be worth more than its pseudo-American value.

To derive a method for valuing American calls, we return to the binomial argument of Section 5-3. With one period remaining before expiration, we suppose the current stock price S will change either to $d(1-\delta)^vS$ or $u(1-\delta)^vS$ by the end of the period; $v = 0$ or 1, depending on whether or not the end of the period is an ex-dividend date. When the call expires, its contract and a rational exercise policy imply that its value must be either $C_u = \max[0, u(1-\delta)^v S - K]$ or $C_d = \max[0, d(1-\delta)^v S - K]$. Therefore,

$$\begin{tikzcd}[column sep=small] & C_u = \max[0, u(1-\delta)^v S - K] \\ C \arrow[ur, dash] \arrow[dr, dash] & \\ & C_d = \max[0, d(1-\delta)^v S - K] \end{tikzcd}$$

Now we can proceed exactly as before. Again we can select a portfolio of Δ shares of stock and the dollar amount B in bonds which will have the same end-of-period value as the call.[32] By retracing our previous steps, we can show that

$$C = [pC_u + (1-p)C_d]/\hat{r}$$

if this is greater than $S - K$, and $C = S - K$ otherwise. Here, once again, $p = (\hat{r} - d)/(u - d)$ and $\Delta = (C_u - C_d)/(u - d)S$.

Thus far the only change is that $(1-\delta)^v S$ has replaced S in the values for C_u and C_d. Now we come to the major difference: Early exercise may be optimal. To see this, suppose that $v = 1$ and $d(1-\delta)S > K$. Since $u > d$, then, also, $u(1-\delta)S > K$. In this case, $C_u = u(1-\delta)S - K$ and $C_d = d(1-\delta)S - K$. Therefore, since $(u/\hat{r})p + (d/\hat{r})(1-p) = 1$,

$$[pC_u + (1-p)C_d]/\hat{r} = (1-\delta)S - (K/\hat{r}).$$

For sufficiently high stock prices, this can obviously be less than $S - K$. Hence, there are definitely some circumstances in which no one would be willing to hold the call for one more period.

In fact, there will always be a critical stock price, $\underline{S}$, such that if $S > \underline{S}$, the call should be exercised immediately. $\underline{S}$ will be the stock price at which $[pC_u + (1-p)C_d]/\hat{r} = S - K$. That is, it is the lowest stock price at which the value of the equivalent portfolio exactly equals $S - K$. This means $\underline{S}$ will, other things equal, be lower the higher the dividend yield, the lower the interest rate, and the lower the striking price.

We can extend the analysis to an arbitrary number of periods in the same way as before. There is only one additional difference, a minor modification in the hedging operation. Now we will buy bonds with any dividends received, and sell bonds to make restitution for dividends paid while we have a short position in the stock.

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[32] Remember that if we are long the portfolio, we will receive the dividend at the end of the period; if we are short, we will have to make restitution for the dividend.

Although the possibility of optimal exercise before the expiration date causes no conceptual difficulties, it does seem to prohibit a simple closed-form solution for the value of a call with many periods to go. However, our analysis suggests a sequential numerical procedure that will allow us to calculate the continuous-time value to any desired degree of accuracy.

Let C be the current value of a call with n periods remaining. Define

$$\bar{v}(n, i) \equiv \sum_{k=1}^{n-i} v_k,$$

so that $\bar{v}(n, i)$ is the number of ex-dividend dates occurring during the next $n - i$ periods. Let $C(n, i, j)$ be the value of the call $n - i$ periods from now, given that the current stock price S has changed to $u^j d^{n-i-j}(1 - \delta)^{\bar{v}(n, i)}S$, where $j = 0, 1, 2, \ldots, n - i$.

, With this notation, we are prepared to solve for the current value of the call by working backward in time from the expiration date. At expiration, $i = 0$, so that

$$C(n, 0, j) = \max[0, u^j d^{n-j}(1 - \delta)^{\bar{v}(n, 0)}S - K] \qquad \text{for} \quad j = 0, 1, \ldots, n.$$

One period before the expiration date, $i = 1$, so that

$$\begin{aligned} C(n, 1, j) = \max\{&u^j d^{n-1-j}(1 - \delta)^{\bar{v}(n, 1)}S - K, \\ &[pC(n, 0, j + 1) + (1 - p)C(n, 0, j)]/\hat{r}\} \\ &\text{for} \quad j = 0, 1, \ldots, n - 1. \end{aligned}$$

More generally, i periods before expiration

$$\begin{aligned} C(n, i, j) = \max\{&u^j d^{n-i-j}(1 - \delta)^{\bar{v}(n, i)}S - K, \\ &[pC(n, i - 1, j + 1) + (1 - p)C(n, i - 1, j)]/\hat{r}\} \\ &\text{for } j = 0, 1, \ldots, n - i. \end{aligned}$$

Observe that each prior step provides the inputs needed to evaluate the right-hand arguments of each succeeding step. The tree diagram in Figure 5-36 illustrates this process. The number of calculations decreases as we move backward in time.

Finally, with n periods before expiration, since $i = n$,

$$C = C(n, n, 0) = \max\{S - K, [pC(n, n - 1, 1) + (1 - p)C(n, n - 1, 0)]/\hat{r}\},$$

and the hedge ratio is

$$\Delta = \frac{C(n, n - 1, 1) - C(n, n - 1, 0)}{(u - d)S}.$$

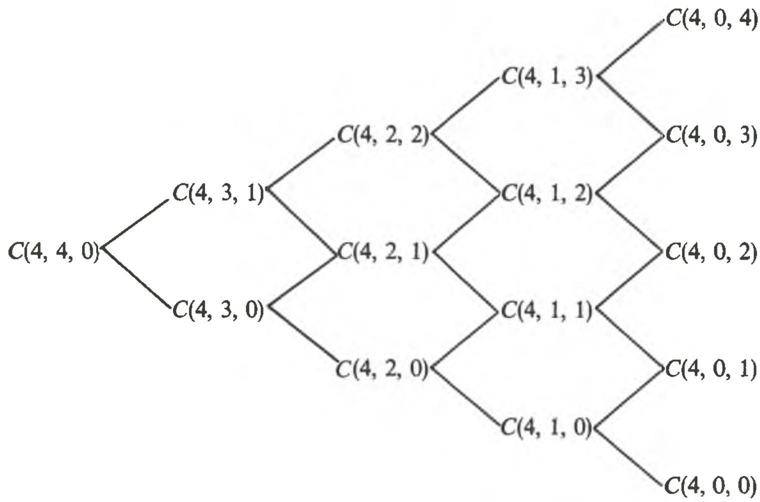


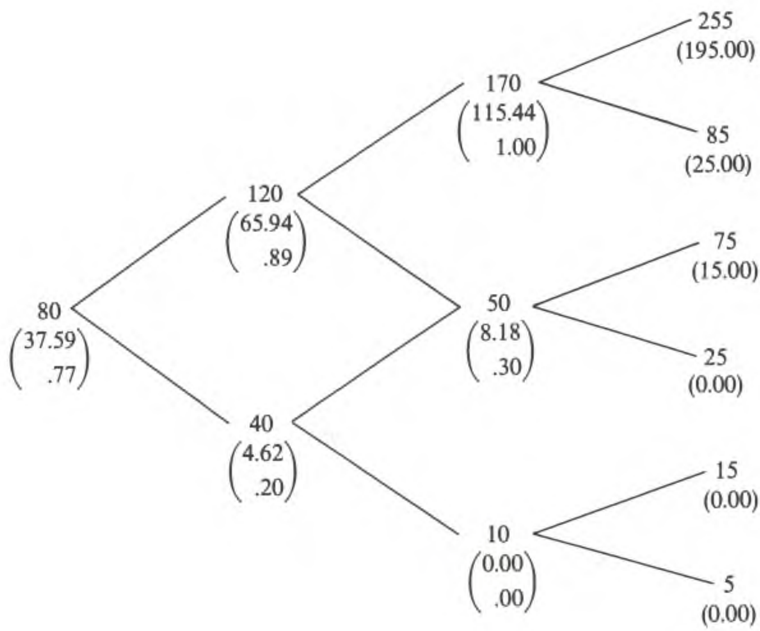
Figure 5-36 Illustration of Binomial Numerical Procedure for Valuing American Options

We could easily expand the analysis to include dividend policies in which the amount paid on any ex-dividend date depends on the stock price at that time in a more general way; the simple case of constant dollar dividends (rather than a constant yield) is especially important.[33] However, this will cause some minor complications. In our present example with a constant dividend yield, the possible stock prices $n - i$ periods from now are completely determined by the total number of upward moves (and ex-dividend dates) occurring during that interval. With other types of dividend policies, the enumeration will be more complicated, since then the terminal stock price will be affected by the timing of the upward moves as well as their total number. But the basic principle remains the same. We go to the expiration date and calculate the call values for all of the possible prices that the stock could have then. Using this information, we step back one period and calculate the call values for all possible stock prices at that time, and so forth.

To illustrate this, consider the case of a constant dollar dividend. Let $n = 3$, $S = 80$, $K = 60$, $r = 1.1$, $u = 1.5$, $d = .5$, and suppose that the stock will pay, when $n = 1$, a dividend of 10 to those who owned the stock during the previous period. Except for the dividend and the different striking price, this is the same as the numerical example given in Section 5-4. Because of the dividend, the stock price when there is one period remaining in the life of the call will be either 170, 50, or 10, rather than 180, 60, or 20 as in the

[33] We could also allow the amount to depend on previous stock prices.

earlier example. The following diagram shows the stock prices, call values, and values of Δ for this example. The last trading time before the stock goes ex-dividend occurs when $n = 2$; at that time, the call is worth more than its exercise value even if the stock price is 120, so early exercise would not be optimal. Note that the stock price can now take on six possible values on the expiration date rather than only four. This shows why a constant dollar dividend will require more computation time than a constant dividend yield.



We will now illustrate the use of the binomial numerical procedure in approximating continuous-time call values. In order to have an exact continuous-time formula to use for comparison, we will consider the case with no dividends. Suppose that we are given the inputs required for the Black-Scholes option pricing formula: S, K, t, σ, and r. To convert this information into the inputs d, u, and $\hat{r}$ required for the binomial numerical procedure, we use the relationships

$$d = 1/u, \qquad u = e^{\sigma\sqrt{t/n}}, \qquad \hat{r} = r^{t/n}.$$

Table 5-3 gives us a feeling for how rapidly option values approximated by the binomial method approach the corresponding limiting Black-Scholes values given by $n = \infty$. Since there are no dividends, the American and European values will be the same. At $n = 5$, the values differ by at most \$.25, and at $n = 20$, they differ by at most \$.07. Although not shown, at $n = 50$ the greatest difference is less than \$.03, and at $n = 150$, the values are identical to the penny.[34]

Table 5-3
BINOMIAL APPROXIMATION OF BLACK-SCHOLES CALL VALUES

$S = 40$ $\quad r = 1.05$

		$n = 5$			$n = 20$			$n = \infty$		
σ	K	*Expiration Month*[a]								
		JAN	*APR*	*JUL*	*JAN*	*APR*	*JUL*	*JAN*	*APR*	*JUL*
	35	5.14	5.77	6.45	5.15	5.77	6.39	5.15	5.76	6.40
.2	40	1.05	2.26	3.12	.99	2.14	2.97	1.00	2.17	3.00
	45	.02	.54	1.15	.02	.51	1.11	.02	.51	1.10
	35	5.21	6.30	7.15	5.22	6.26	7.19	5.22	6.25	7.17
.3	40	1.53	3.21	4.36	1.44	3.04	4.14	1.46	3.07	4.19
	45	.11	1.28	2.12	.15	1.28	2.23	.16	1.25	2.24
	35	5.40	6.87	7.92	5.39	6.91	8.05	5.39	6.89	8.09
.4	40	2.01	4.16	5.61	1.90	3.93	5.31	1.92	3.98	5.37
	45	.46	1.99	3.30	.42	2.09	3.42	.42	2.10	3.43

NOTE: Assumes no dividends will be paid during the lives of the options.
[a] The January options have one month to expiration; the Aprils, four months; and the Julys, seven months. Both r and σ are expressed in annual terms.

Having checked the accuracy of the binomial numerical procedure on a problem for which we already know the answer, we are prepared to apply it to a new problem—the effect of cash dividends on call values. Now we also need to know δ and the integers k for which $v_k = 1$. Given t and n, these are the integral values of k for which $(k/n)t$ is approximately an ex-dividend date. Consider circumstances identical to those in Table 5-3, except that the stock has a quarterly dividend yield of $\delta = .0125$, which on

[34] Since the Black-Scholes values ($n = \infty$) were calculated directly from the Black-Scholes formula, these results demonstrate that our binomial numerical procedure is correct. They also give some indication of its efficiency (that is, accuracy versus computer time). The option values in Tables 5-2 through 5-6 were calculated on an IBM 5100 portable computer, using APL, an interactive computer language.

the basis of the current stock price of \$40 would correspond to a quarterly dividend of \$.50. Additionally, suppose the first quarterly dividend is due in one-half month from the present. Table 5-4 provides European, pseudo-American, and American call values under these conditions.

Table 5-4
BLACK-SCHOLES CALL VALUES WITH CASH DIVIDENDS

$S = 40 \quad D = (2, \frac{1}{2})^a \quad r = 1.05$

		European			*Pseudo-American*			*American* ($n = 150$)		
σ	K	*Expiration Month*[b]								
		JAN	*APR*	*JUL*	*JAN*	*APR*	*JUL*	*JAN*	*APR*	*JUL*
.2	35	4.66	4.88	5.16	5.07[c]	5.19[c]	5.44[c]	5.07	5.23	5.51
	40	.76	1.64	2.19	.76	1.73[c]	2.31[c]	.83	1.79	2.36
	45	.01	.33	.72	.01	.33	.74[c]	.01	.36	.78
.3	35	4.75	5.46	6.04	5.08[c]	5.67[c]	6.25[c]	5.10	5.74	6.34
	40	1.21	2.54	3.36	1.21	2.58[c]	3.45[c]	1.27	2.69	3.54
	45	.12	.98	1.71	.12	.98	1.72[c]	.12	1.03	1.80
.4	35	4.95	6.15	7.02	5.13[c]	6.28[c]	7.17[c]	5.22	6.40	7.29
	40	1.67	3.44	4.53	1.67	3.44	4.59[c]	1.72	3.59	4.72
	45	.34	1.76	2.81	.34	1.76	2.81	.34	1.83	2.92

NOTE. The European values are adjusted for dividends by replacing the current stock price (that is, \$40) by $S(1-\delta)^{\bar{v}}$, where $\bar{v}$ is the number of ex-dividend dates prior to an option's expiration date, and δ is the current indicated quarterly dividend yield (that is, .5/40). For each option, the pseudo-American values adjust for the dividend by selecting the highest of at most four European values, one for early exercise just prior to each ex-dividend date prior to expiration, and one assuming exercise at expiration. The American values are adjusted for dividends using the binomial numerical approximation procedure and therefore give full consideration to the impact of early exercise.

[a] $D = (2, \frac{1}{2})$ signifies that the indicated annual dividend, based on the current price and yield, is \$2, and that the first quarterly ex-dividend date from the present occurs in half a month. Both r and σ are expressed in annual terms.

[b] The January options have one month to expiration; the Aprils, four months; and the Julys, seven months.

[c] Early exercise is indicated. In every case, this occurs on the last ex-dividend date prior to the call's expiration.

By comparing the limiting Black-Scholes values in Table 5-3 with the European values in Table 5-4, we see that a dividend yield of 5% per year has a significant impact on call values. As we would expect, the effect of dividends is smaller if we allow for the possibility of early exercise. Even the limited flexibility represented in the pseudo-American values gives a sizeable increase for many of the calls, particularly those that are in-the-money.

Early exercise on the last ex-dividend date prior to expiration is indicated for all of the in-the-money calls, as well as some that are now at-the-money or even out-of-the-money. Of course, where early exercise is not indicated, European and pseudo-American values are the same.

As we have previously mentioned, the pseudo-American correction is not complete. Not surprisingly, all American values are somewhat higher, since they reflect the full flexibility to the call buyer from the privilege of early exercise. However, in some cases, the difference is so small that it is eliminated by rounding to the nearest penny, as we have done in the tables. Perhaps the most striking feature shown in Tables 5-3 and 5-4 is the difference between the American call values with and without dividends.[35] One might have thought the possibility of early exercise would mean that dividends would have only a small effect on the value of an American call. However, this opportunity comes only at the cost of voluntarily ending the life of a call that may have many months remaining before expiration and paying the striking price immediately rather than later. Correspondingly, we find that the differences in value are not only significant, but also increase with the length of maturity of the call.

We have used a constant dividend yield in most of our examples only to make the exposition easier. Firms often try to maintain a constant dividend yield in the long run, but not in the short run. Consequently, a constant dividend yield may be a satisfactory way to represent dividends for an option with several years until expiration, while a constant dollar dividend would be more appropriate for the shorter lifetimes of listed options. As we have seen, the binomial numerical method can easily include constant dollar dividends. Calculations with a constant quarterly dividend of $.50 gave values that were virtually identical to those shown in Table 5-4. This suggests that even in those situations where constant dollar dividends are the appropriate choice, a constant dividend yield will still give very useful results if the yield parameter is continually readjusted as the stock price changes in order to keep the implied dividends equal to the constant amount.

American Puts. To derive a method for valuing puts, we again use the binomial formulation. Although it has been convenient to express the argument in terms of a particular security, a call, this is not essential in any way. The same basic analysis can be applied to puts.

[35] Since a payout-protected option would have the same value for any dividend policy as it would if the stock pays no dividends, this comparison also shows the value of payout protection. Note that the Black-Scholes model satisfies the conditions for the payout-protection adjustment discussed in Section 4-4. The distribution of the rate of return with reinvestment of cash dividends does not depend on the firm's dividend policy.

Letting P denote the current price of a put, with one period remaining before expiration we have

$$P \begin{cases} P_u = \max[0,\ K - u(1-\delta)^v S] \\ \\ P_d = \max[0,\ K - d(1-\delta)^v S]. \end{cases}$$

Once again, we can choose a portfolio of $S\Delta$ in stock and B in bonds which will have the same end-of-period value as the put. By a series of steps which are formally equivalent to the ones we followed in Section 5-3, we can show that

$$P = [pP_u + (1-p)P_d]/\hat{r},$$

if this is greater than $K - S$, and $P = K - S$ otherwise. As before, $p = (\hat{r} - d)/(u - d)$ and $\Delta = (P_u - P_d)/(u - d)S$. Note that for puts, since $P_u \leq P_d$, then $\Delta \leq 0$. This means that if we sell an overvalued put, the equivalent portfolio that we buy will involve a short position in the stock.

We might hope that with puts we will be spared the complications caused by optimal exercise before the expiration date. Unfortunately, this is not the case. In fact, the situation is even worse in this regard. Now there are always some possible circumstances in which no one would be willing to hold the put for one more period.

To see this, suppose $K > u(1-\delta)^v S$. Since $u > d$, then, also, $K > d(1-\delta)^v S$. In this case, $P_u = K - u(1-\delta)^v S$ and $P_d = K - d(1-\delta)^v S$. Therefore, since $(u/\hat{r})p + (d/\hat{r})(1-p) = 1$,

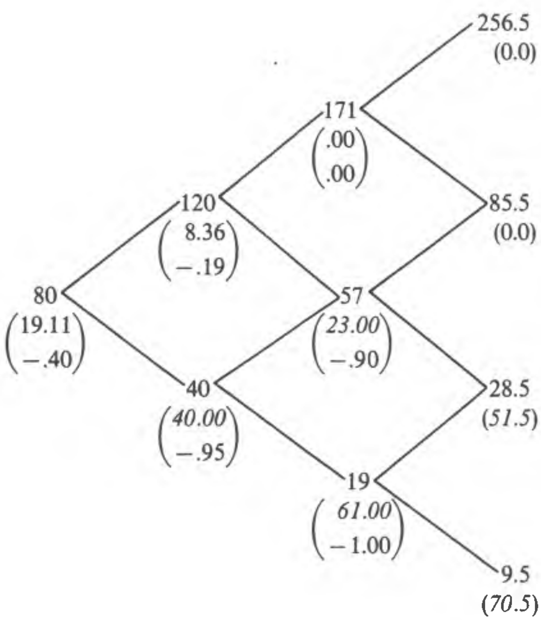
$$[pP_u + (1-p)P_d]/\hat{r} = (K/\hat{r}) - (1-\delta)^v S.$$

If there are no dividends (that is, $v = 0$), then this is certainly less than $K - S$. Even with $v = 1$, it will be less for a sufficiently low stock price.

Thus, there will now be a critical stock price, $\bar{S}$, such that if $S < \bar{S}$, the put should be exercised immediately. By analogy with our discussion for the call, we can see that this is the stock price at which $[pP_u + (1-p)P_d]/\hat{r} = K - S$. Other things equal, $\bar{S}$ will be higher the lower the dividend yield, the higher the interest rate, and the higher the striking price. Optimal early exercise thus becomes more likely if the put is deep-in-the-money and the interest rate is high. The effect of dividends yet to be paid diminishes the advantages of immediate exercise, since the put buyer will be reluctant to sacrifice the forced declines in the stock price on future ex-dividend dates.

This argument can be extended in the same way as before to value puts with any number of periods to go. However, the chance for optimal exercise before the expiration date once again seems to preclude the possibility of expressing this value in a simple form. But our analysis also indicates that, with slight modification, we can value puts with the same numerical techniques we use for calls. Reversing the difference between the stock price and the striking price at each stage is the only change.[36]

The following diagram shows the stock prices, put values, and values of Δ obtained in this way for the example given in Section 5-4. The values used there were $S = 80$, $K = 80$, $n = 3$, $u = 1.5$, $d = .5$, and $\hat{r} = 1.1$. To include dividends as well, we assume that a cash dividend of 5% ($\delta = .05$)



[36] Given the theoretical basis for the binomial numerical procedure provided, the numerical method can be generalized to permit $k + 1 \leq n$ jumps to new stock prices in each period. We can consider exercise only every k periods, using the binomial formula to leap across intermediate periods. In effect, this means permitting $k + 1$ possible new stock prices before exercise is again considered. That is, instead of considering exercise n times, we would only consider it about n/k times. For fixed t and k, as $n \rightarrow \infty$, option values will approach their continuous-time values.

This alternative procedure is interesting, since it may enhance computer efficiency. At one extreme, for calls on stocks that do not pay dividends, setting $k + 1 = n$ gives the most efficient results. However, when the effect of potential early exercise is important and greater accuracy is required, the most efficient results are achieved by setting $k = 1$, as in our description above.

will be paid at the end of the last period before the expiration date. Thus, $(1-\delta)^{v(n,0)} = .95$, $(1-\delta)^{v(n,1)} = .95$, and $(1-\delta)^{v(n,2)} = 1.0$. Put values in italics indicate that immediate exercise is optimal.

Table 5-5 contrasts the values of otherwise identical European and American puts on a stock that does not pay cash dividends prior to the expiration date. For many puts, particularly the low-volatility, long-maturity, deep-in-the-money puts, potential early exercise significantly affects their value. Indeed, since the low-volatility, in-the-money put with one month to expiration is selling at its parity value (that is, $K - S$), immediate exercise is advisable. Incidentally, observe that the low-volatility, in-the-money European puts do not necessarily increase in value as their maturity lengthens.

A comparison of Tables 5-5 and 5-6 shows that dividends increase the value of unprotected European and American puts, just as we would expect. It is interesting to note that the increase is usually larger for a European put than for the corresponding American put. This is because the owner of the European put will get the benefits of the decreases in the stock price on all of the ex-dividend dates, while the owner of the American put may find it advantageous to forego some of these benefits through early exercise in order to receive the striking price sooner. As a result, cash dividends tend to reduce the difference between the values of European and American puts.

Table 5-5
REPRESENTATIVE BLACK-SCHOLES PUT VALUES

		$S = 40$		$r = 1.05$			
		European			*American* ($n = 150$)		
σ	K	*Expiration Month*[a]					
		JAN	*APR*	*JUL*	*JAN*	*APR*	*JUL*
.2	35	.01	.20	.42	.01	.20	.43
	40	.84	1.52	1.88	.85	1.58	1.99
	45	4.84	4.78	4.84	*5.00*[b]	5.09	5.27
.3	35	.08	.69	1.19	.08	.70	1.22
	40	1.30	2.43	3.06	1.31	2.48	3.17
	45	4.98	5.53	5.97	5.06	5.71	6.24
.4	35	.25	1.33	2.11	.25	1.35	2.16
	40	1.76	3.33	4.25	1.77	3.38	4.35
	45	5.24	6.38	7.17	5.29	6.51	7.39

NOTE: Assumes no dividends will be paid during the lives of the options.

[a] The January options have one month to expiration; the Aprils, four months; and the Julys, seven months. Both r and σ are expressed in annual terms.

[b] Exercise immediately.

Table 5-6
BLACK-SCHOLES PUT VALUES WITH CASH DIVIDENDS

$S = 40 \;\; D = (2, \tfrac{1}{2}) \;\; r = 1.05$

		European			*American* ($n = 150$)		
σ	K	*Expiration Month*[a]					
		JAN	*APR*	*JUL*	*JAN*	*APR*	*JUL*
.2	35	.01	.30	.65	.01	.31	.66
	40	1.09	1.98	2.54	1.11	2.01	2.58
	45	5.33	5.60	5.93	5.41	5.67	6.02
.3	35	.11	.88	1.53	.11	.88	1.55
	40	1.55	2.88	3.71	1.56	2.91	3.74
	45	5.43	6.24	6.92	5.50	6.29	6.99
.4	35	.30	1.57	2.51	.31	1.58	2.52
	40	2.00	3.78	4.88	2.01	3.81	4.92
	45	5.65	7.02	8.02	5.70	7.07	8.10

NOTE: The European values are adjusted for cash dividends by replacing the current stock price (that is, \$40) by $S(1 - \delta)^{\bar{v}}$, where $\bar{v}$ is the number of ex-dividend dates prior to an option's expiration and δ is the current indicated quarterly dividend yield (that is, .5/40). The American values are adjusted for dividends using the binomial numerical approximation procedure and therefore give full consideration to the impact of early exercise.

[a] The January options have one month to expiration; the Aprils, four months; and the Julys, seven months. Both r and σ are expressed in annual terms.

Figure 5-37 is a graph of the optimal exercise boundary with and without dividends for two otherwise identical unprotected American puts. It shows how the boundary changes for a given put as the expiration date comes closer. The upper curve, with no dividends, shows that the boundary steadily increases as time passes. For example, assume that the annualized interest rate is .05, the striking price is \$40, and the stock's annual volatility is .3. With nine months to expiration, the current stock price can be as low as \$28.80, or 72% of K, without causing immediate exercise to be desirable. However, exercise is optimal at any lower stock price. Eight months later, when the put has one month to expiration, immediate exercise is advisable if the stock is below \$34.28, or 85.7% of K. Further calculation shows that the exercise boundary rises with a lower volatility or a higher interest rate. For example, if $\sigma = .2$, other things equal, the optimal exercise boundary is again a smooth convex curve falling from \$40 at $t = 0$ to \$36 at $t = .75$.

In Section 4-2, we remarked that the optimal exercise boundary for unprotected American puts on stocks which pay dividends may behave in the following way as time passes. The boundary drops to zero just before an ex-dividend date, and then immediately after the ex-dividend date, it jumps

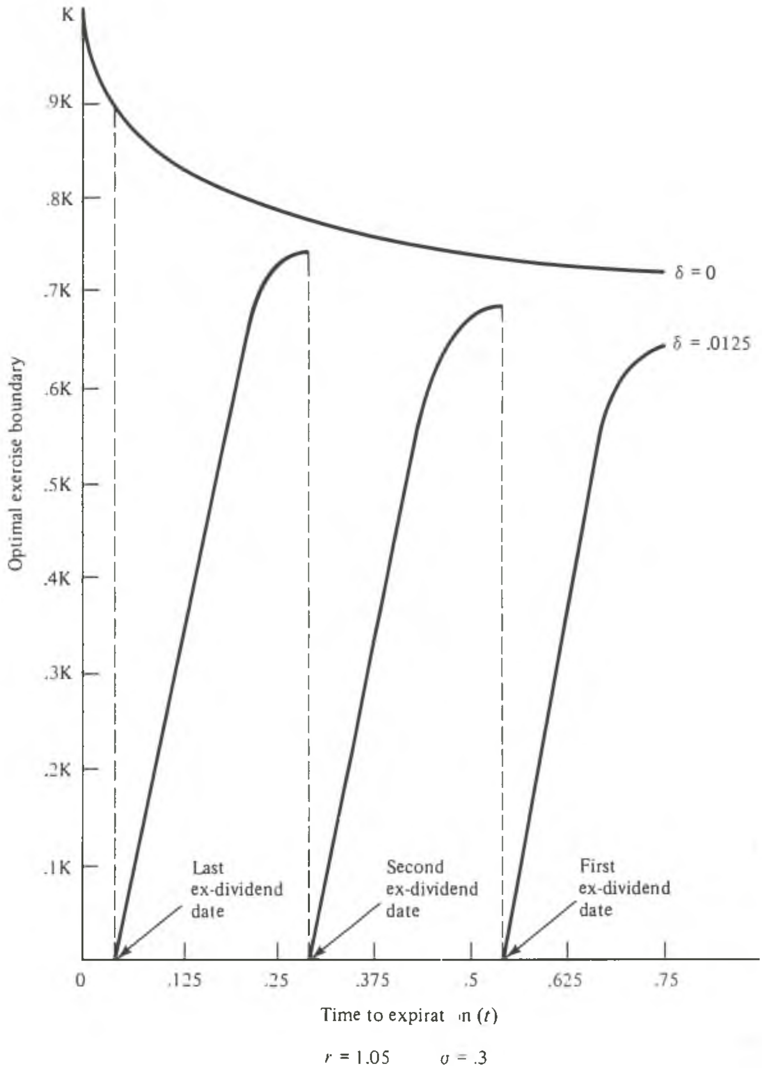


Figure 5-37 Optimal Exercise Boundary for an American Put

upward to a level greater than it had just after the previous ex-dividend date. If this is not the last ex-dividend date, the boundary then starts declining to zero. If it is the last one, the boundary begins to increase and rises to K as the time to expiration goes to zero. Figure 5-37 not only confirms this behavior, but allows us to make two more specific observations for our sample put. First, prior to the last ex-dividend date, the optimal exercise boundary for the put with dividends lies everywhere below the boundary for the otherwise identical put without dividends. Second, between ex-dividend dates the exercise boundary is concave to the origin. The corresponding graph for the case of a constant dollar dividend of \$.50 per quarter is very similar to Figure 5-37. The main differences are that after each ex-dividend date the boundary drops much more rapidly to nearly zero and then has a convex segment which reaches zero just before the next ex-dividend date.

It may seem puzzling that the optimal boundary ever goes to zero, but there is an intuitive explanation. Essentially, this says that no matter how low the stock price is, you should never exercise a put immediately before an ex-dividend date. Here is why. Let S be the stock price one second before the ex-dividend date, S^x be the stock price one second afterward, and D be the amount of the dividend. Since other changes in the stock price will be extremely small over so short a period, S^x will be approximately $S - D$. By postponing exercise for two seconds, you get $K - S^x = K - (S - D)$ instead of $K - S$. Of course, you would lose two seconds' worth of interest on $K - S$, but this will be completely insignificant relative to the amount of the dividend. Consequently, even if only a very small dividend will be paid, a put should never be exercised just before an ex-dividend date.

To contrast this with an otherwise identical unprotected American call, calculation confirms that it only pays to consider early exercise just before an ex-dividend date. In effect, the optimal exercise boundary is infinite between these dates. On the first ex-dividend date the stock price must be greater than about \$56.80, or 142% of K, for exercise to be optimal. On the second ex-dividend date the critical stock price is \$51.60, and on the last it decreases to \$42. Additional calculation shows that an increase in the dividend yield decreases these critical values, while an increase in volatility or the interest rate increases them.

Table 5-7 summarizes many of our results about the optimal exercise boundary. Each entry answers the following question. Suppose that in Figure 5-37 we simultaneously draw graphs for two sets of inputs which are the same except for one variable. Will the graph corresponding to the higher value of that variable lie above or below the graph for the lower value of the variable? In the table, an upward arrow means that it will always lie above, while a downward arrow means it will always lie below. Of course, for options with different expiration dates, this comparison would be defined only for the period in which both are outstanding; at the point on the horizontal axis corresponding to the expiration date of the shorter maturity option, the longer one would still have some time remaining. Here is an easy way to remember these results: For a change in any variable except the striking price, the value of an option and its optimal exercise boundary will shift in the same direction for a call and in the opposite direction for a put.

Table 5-7
THE EFFECT OF CHANGES IN THE UNDERLYING VARIABLES ON THE OPTIMAL EXERCISE BOUNDARY

Determining Factors	*Effect of Increase*	
	Put	*Call*
Striking price	↑	↑
Expiration date	↓	↑
Volatility	↓	↑
Interest rate	↑	↑
Cash dividends	↓	↓

Finally, we wish to urge caution in making such statements as "a put with a high volatility is more likely to be exercised during the next month than an otherwise identical put with a low volatility." In the first place, the actual probability that optimal early exercise will occur during any given time period depends on the expected rate of return on the stock, which was not necessary in calculating the optimal boundary or the option value. Furthermore, even if we use the same expected rate of return in each case, the comparison is by no means simple. A higher volatility would indeed make it more likely that the stock price will reach a *fixed* low level, but it will simultaneously lower the level which must be reached. The final result depends on which effect dominates. The same issues arise in comparisons involving different dividend policies. This kind of problem can be solved, but it is relatively complicated, so we will not pursue it further.