Class 15 Curvature of cov der

(Chap 12)

1) Def. $C^\infty(M;E)$ is the space the section on v.b. E.

∇ cov der

$\nabla : C^\infty(M;E) \to C^\infty(M; E \otimes T^*M)$

1) cov. der exists.

2) ∇, ∇' are cov der

iff $\underline{\nabla - \nabla' \text{ is a section of } \underline{\underline{\mathrm{End}(E) \otimes T^*M}}}$

Rem. 1) ∇ is not a section.

2) we consider sections of $\mathrm{End}(E) \otimes T^*M$.

as matrix valued 1-form. α

$$\alpha = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \alpha_{n1} & & \alpha_{nn} \end{bmatrix}$$ α_{ij} are 1-forms.

T^*M. locally. $dx^1, \cdots, dx^n$.

$$\alpha = \underbrace{\begin{bmatrix} & \end{bmatrix}}_{A_1} dx^1 + \cdots \underbrace{\begin{bmatrix} & \end{bmatrix}}_{A_n} dx^n$$

matrix-value function (0-form).

$$\alpha s = \underline{\sum (A_i s) dx^i}$$ s section of E

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ & & \end{bmatrix} \wedge \begin{bmatrix} \beta_{11} & \\ \vdots & \\ \beta_{n1} & \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} \wedge \beta_{11} + \alpha_{12} \wedge \beta_{21} + \cdots \\ \\ \end{bmatrix}$$

exterior derivative -

$\Omega^k = C^\infty(M; \underline{\Lambda^k T^*M})$ sections of k-forms.

$\underline{d} : \Omega^k \to \Omega^{k+1}$ s.t. $d^2 = 0$.

($H_{dR} = \ker d / \operatorname{Im} d$)

1) $d(w_1 + w_2) = dw_1 + dw_2$

2) $d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$

sign.

We combine ∇ d together to define exterior cov. der.

$d_\nabla : \underline{C^\infty(E \otimes \Lambda^k T^*M) \to C^\infty(E \otimes \Lambda^{k+1} T^*M)}$

$k=0$. $\Lambda^0 T^*M = M \times \mathbb{R}$.

$$C^\infty(E) \longrightarrow C^\infty(E \otimes \Lambda^1 T^*M)$$

$$d_\nabla s = \nabla s$$

$k>0$. a section in $C^\infty(E \otimes \Lambda^k T^*M)$

can be written as $\sum s \otimes \omega$.

$s \in C^\infty(E)$ $\omega \in C^\infty(\Lambda^k T^*M)$

$$\underline{d_\nabla(s \otimes \omega)} = \underline{\nabla s \wedge \omega + s \otimes d\omega}$$

as Leibniz rule

$$\nabla s \otimes \omega + s \otimes d\omega$$

$\underline{E \otimes T^*M \otimes \Lambda^k T^*M}$ is not well-defined section

in $C^\infty(E \otimes \Lambda^{k+1} T^*M)$ $\neq$

∇. E can be extended to

E^*, $E \otimes E$

$$d_\nabla\left(\sum_i s_i \otimes w_i\right) = \sum_i d_\nabla(s_i \otimes w_i)$$

$d^2 = 0$ $\quad d^2 : C^\infty(\Lambda^k T^* M) \to C^\infty(\Lambda^{k+2} T^* M)$

$d_\nabla^2 : C^\infty(E \otimes \Lambda^k T^* M) \to C^\infty(E \otimes \Lambda^{k+2} T^* M)$

(if $d_\nabla^2 = 0$, then maybe we can define something similar to dR cohomology)

Rem: $d_\nabla^2 \neq 0$ in general.

For $s \in C^\infty(M; E)$, define

$$d_\nabla^2 s = F_\nabla s$$

where F_∇ is called curvature of ∇

Later. we will show it is a

matrix valued 2-form

$d_\nabla^2 s \in C^\infty(E \otimes \Lambda^2 T^*M)$.

$F_\nabla \in C^\infty(\mathrm{End}(E) \otimes \Lambda^2 T^*M)$.

$\nabla - \nabla' \in C^\infty(\mathrm{End}(E) \otimes \Lambda^1 T^*M)$

we prove this claim first.

Lem. if $S: E \to E'$ is linear over $C^\infty(M;\mathbb{R}) \ni f$ i.e. $fs = sf$

$\Rightarrow$ S is a sec'n of $\mathrm{Hom}(E, E')$

Last time. $(\nabla - \nabla')(fs) = f(\nabla - \nabla')s$

Now. we need to check

$$F_\nabla(fs) = f(F_\nabla s).$$

$$F_\nabla(fs) = d_\nabla^2(fs) = d_\nabla(d_\nabla(fs))$$

$$= d_\nabla(f d_\nabla s + s \otimes df)$$

$$= d_\nabla(f d_\nabla s) + d_\nabla(s \otimes df)$$

$$= df \wedge d_\nabla s + f d_\nabla^2 s$$

$$+ d_\nabla s \wedge df + s \otimes \cancel{d^2 f}^{\,0}$$

$df \wedge d_\nabla s = - d_\nabla s \wedge df$ because df is 1 form. $d_\nabla s$ is 1-form.

$$= f d_\nabla^2 s = f(F_\nabla s)$$

$$d_\nabla^2(s\otimes w) = \boxed{F_\nabla s} \wedge w.$$

w k-form. not just 0-form f

$$d_\nabla(d_\nabla s\wedge w + s\otimes dw)$$

$$= \boxed{d_\nabla^2 s\wedge w} - \cancel{d_\nabla s \wedge dw} + \cancel{d_\nabla s\wedge dw} + \cancel{s\otimes d^2 w}$$

0

$d_\nabla s$ is a "1-form"

$E\otimes\Lambda^1 T^*M$

"E-valued" 1-form.

α_i: 1-form

e_i is basis of E

$$d_\nabla s = e_1\otimes\alpha_1 + e_2\otimes\alpha_2 + \dots$$

$$d_\nabla s\wedge w = e_1\otimes(\alpha_1\wedge w) + e_2\otimes(\alpha_2\wedge w)$$

$$\underbrace{F_\nabla s \wedge \omega}_{\parallel} \neq \underbrace{F_\nabla (s \wedge \omega)}_{\not\exists}$$

$d_\nabla^2 s \wedge \omega.$ no def.

s sect of E

ω k-form do not have $s \wedge \omega$.

$\nabla - \nabla^0 = \alpha \Rightarrow \nabla = \nabla^0 + \alpha$

Locally, we write. $\phi_U : U \to \mathbb{R}^m$.

$\varphi_U : E|_U \to U \times \mathbb{R}^n$. $\dim M = m$

$\dim E|_p = n.$

$s : M \to E$. $s_U = \phi_U^{-1} \circ s : \mathbb{R}^n \to \mathbb{R}$

$\varphi_U(\nabla^0 s(p)) = (p, d s_U(p))$

$dsu(p)$:

$su(p) = (su_1(p), \dots, su_n(p))$

$su_i(p) : \mathbb{R} \to \mathbb{R}$

$(su_i(p))_* : T\mathbb{R} \to T\mathbb{R} = \mathbb{R}$

a 1-form

dsu is a section of $E \otimes T^*M|_U$

$\varphi_u(\nabla s)(p) = (p, dsu(p) + \alpha_u(p) su(p))$

α_u : matrix-valued 1-form of U.

$(\mathrm{End}(E) \otimes \Lambda^1 T^*M)|_U$

Local formula for F_∇

$$F_\nabla s = d_\nabla^2 s = d_\nabla(d_\nabla s)$$

$$= d_\nabla(\nabla s)$$

$$\varphi_u(d_\nabla^2 s)(p)$$

$$= (p, \; d(\; \nabla s \;) + \alpha_u(\; \nabla s \;))$$

$$= (p, d(ds_u + \alpha_u s_u) + \alpha_u(ds_u + \alpha_u s_u))$$

$$= (p, d^2 s_u + d(\alpha_u s_u) + \alpha_u \wedge ds_u + \underline{\alpha_u \wedge \alpha_u s_u}$$

$$= (p, \cancelto{0}{d^2 s_u} + (d\alpha_u) s_u - \cancel{\alpha_u \wedge ds_u}$$

$(d(\omega_1 \wedge \omega_2)$ α_u 1-form$)$

$$+ \cancel{\alpha_u \wedge ds_u} + (\alpha_u \wedge \alpha_u) s_u$$

$\varphi_u(d^2_\nabla s)(p)$

$= (p, (d\alpha_u + \alpha_u \wedge \alpha_u) s_u)$

$(F_\nabla)|_u s_u.$

$$F_\nabla|_u = \underline{d\alpha_u + \alpha_u \wedge \alpha_u}.$$

α_u matrix valued 1-form

$d\alpha_u$ . . 2 form.

$\alpha_u \wedge \alpha_u$ 2-form

matrix of functions

Notation : $\alpha_u = \sum_k \alpha_{u_k} dx^k$

$$\alpha_u \wedge \alpha_u = \sum_{i,j} \underline{\alpha_{u_i} \alpha_{u_j}}\, dx^i \wedge dx^j$$

$(dx^i \wedge dx^j) = -dx^j \wedge dx^i$

$$= \sum_{i<j} (\alpha_{u_i} \alpha_{u_j} - \alpha_{u_j} \alpha_{u_i})\, dx^i \wedge dx^j$$

$$[\alpha_{u_i}, \alpha_{u_j}] = \underline{\alpha_{u_i}\alpha_{u_j} - \alpha_{u_j}\alpha_{u_i}}$$

$$(F_\nabla)_u = \underline{\underline{\sum_{i<j}}} \boxed{\left(\partial_i \alpha_{u_j} - \partial_j \alpha_{u_i} + [\alpha_{u_i}, \alpha_{u_j}] \right) dx^i \wedge dx^j}$$

$$(F_\nabla)_{u_{ij}} = \partial_i \alpha_{u_j} - \partial_j \alpha_{u_i} + [\alpha_{u_i}, \alpha_{u_j}]$$

$$(F_\nabla)_v = d\alpha_v + \alpha_v \wedge \alpha_v$$

$$\alpha_v = g_{uv}^{-1} \alpha_u g_{uv} + g_{uv}^{-1} d g_{uv}$$

$$= g_{vu} \alpha_u g_{vu}^{-1} + g_{vu} d g_{vu}^{-1}$$

$$\Rightarrow (F_\nabla)_u = g_{uv}^{-1} (F_\nabla)_v \, g_{uv}$$

F_∇ is a section of

$E \otimes \Lambda^2 T^*M.$

Meaning of F_∇ (in a local chart)

$dx^1, \ldots, dx^n$ as basis of $T^*M|_u$.

$$\nabla s_u = \sum_k \nabla_k s_u \, dx^k.$$

$$\nabla_k s_u = \nabla_{\frac{\partial}{\partial x^k}} s_u.$$

$$\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$$

$$\nabla_k : C^\infty(E) \to C^\infty(E)$$ not a section End(E)

Recall for usual derivative.

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

Rem. $\frac{\partial}{\partial x^i}$ corresponds to $\nabla^0_{\frac{\partial}{\partial x^i}}$

$$\nabla_i \nabla_j Su - \nabla_j \nabla_i Su.$$

$$\nabla^0_{\frac{\partial}{\partial x^k}} = \frac{\partial}{\partial x^k}. \qquad \nabla = \nabla^0 + \alpha.$$

$$\nabla_i Su = \partial_i Su + \alpha_i Su.$$

$$\nabla_i \nabla_j Su = \partial_i(\qquad) + \alpha_i(\qquad)$$

$$= \partial_i(\partial_j Su + \alpha_j Su) + \alpha_i(\partial_j Su + \alpha_j Su)$$

$$= \partial_i \partial_j Su + \partial_i(\alpha_j Su) + \alpha_i \partial_j Su + \alpha_i \alpha_j Su$$

$$= \underline{\cancel{\partial_i \partial_j Su}} + (\partial_i \alpha_j) Su + \cancel{\alpha_j \partial_i Su}$$
$$+ \cancel{\alpha_i \partial_j Su} + \alpha_i \alpha_j Su.$$

$\nabla_j \nabla_i Su =$

$$\underline{\cancel{\partial_j \partial_i Su}} + (\partial_j \alpha_i) Su + \cancel{\alpha_i \partial_j Su}$$
$$+ \cancel{\alpha_j \partial_i Su} + \alpha_j \alpha_i Su$$

$(\nabla_i \nabla_j - \nabla_j \nabla_i) Su$

$$= (\partial_i \alpha_j + \alpha_i \alpha_j - \partial_j \alpha_i - \alpha_j \alpha_i) Su$$
$$= (\partial_i \alpha_j - \partial_j \alpha_i + \underline{\alpha_i \alpha_j - \alpha_j \alpha_i}) Su$$

Rem. α_i, α_j matrices. $\alpha_i \alpha_j \neq \alpha_j \alpha_i$ in general.

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) s_u = (F_{\nabla ij})_u s_u$$

$$(F_\nabla)_u = \sum_{i<j} (F_{\nabla ij})_u \, dx^i \wedge dx^j$$

$$= \frac{1}{2} \sum_{i,j} (F_{\nabla ij})_u \, dx^i \wedge dx^j$$

① $d_\nabla^2 \neq 0$

② $\nabla_i \nabla_j - \nabla_j \nabla_i \neq 0$

F_∇ captures information here