MATH 230A Assignment 3

Problem 1:

Let $H = \{(x,y) \in \mathbb{R}^2 \,\big|\, y > 0\}$ be the half-plane and define the Riemanian metric on H by

$$g_0 = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

(This is usually called the **hyperbolic metric**. Let $D = \{(x,y) \in \mathbb{R}^2 \,\big|\, x^2 + y^2 < 1\}$ and let $z = x + iy \in \mathbb{C} = \mathbb{R}^2$. Define the map $f_0 : D \to H$ by

$$f_0(z) = i \cdot \frac{1-z}{1+z}.$$

1. Prove f_0 is a diffeomorphism between D and H.
2. Suppose $f : M \to N$ is a smooth map between manifolds and g is a metric on N. Suppose $f(p_0) = q_0$. Let $U \subset M$ be a chart of p_0 with $\phi_U : U \to \mathbb{R}^m$ and let $V \subset N$ be a chart of q_0 with $\phi_V : V \to \mathbb{R}^n$. In the chart V, let
$$g = \phi_U^*(g_{ij} \cdot dx^i \otimes dx^j),$$
where $g_{ij} : V \to \mathbb{R}$ is a smooth map. Compute the pull-back metric f^*g on the chart $f^{-1}(V) \cap U$.
3. Compute the pull-back metric $f_0^* g_0$ on D.

Problem 2: Let $T \subset \mathbb{R}^3$ be obtained by rotating the circle

$$\{(x,y,z) \,\big|\, y = 0, z^2 + (x-2)^2 = 1\}$$

about the z-axis. Let T be parameterized by the coordinates $(\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$

$$f(\theta, \phi) = ((2 + \sin\phi)\cos\theta, (2 + \sin\phi)\sin\theta, \cos\phi).$$

Define the Riemannian metric g_0 on T by the induced metric from $\mathbb{R}^3$. Define the metric g on $S^1 \times S^1$ by the pull-back metric $f^* g_0$.

1. Prove that f is $1-1$ and onto as a map from $S^1 \times S^1$ to T such that its differential as map into $\mathbb{R}^3$ is injective. (This implies f is a diffeomorphism.)
2. Compute the inner product $g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}), g(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}), g(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi})$, and the pull-back metric g.

Problem 3: Let $U \subset M$ be a chart with $\phi_U : U \to \mathbb{R}^n$. Suppose $\gamma : I = [0,1] \to M$ is a smooth map (called a **path**). Let

$$\phi_U \circ \gamma : \gamma^{-1}(U) \subset I \to \mathbb{R}^n$$

be denoted by

$$(\gamma^1(t), \ldots, \gamma^n(t))$$

and let

$$\dot\gamma^i = \frac{d\gamma^i}{dt}, \ \ddot\gamma^i = \frac{d^2\gamma^i}{dt}.$$

The geodesic equation is

$$\ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0,$$

where the indice that appear twice should be taken a sum on $1, \ldots, n$, *i.e.* $\Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = \sum_{j,k=1}^n \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k$. Note that the **Christoffel symbol**

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}),$$

where $\{g^{il}\}_{i,l\in[1,n]}$ is the inverse matrix of $\{g_{il}\}_{i,l\in[1,n]}$.

Prove that the solution of the geodesic equation is independent of the chart, *i.e.* given another chart $V \subset M$ with $U \cap V \neq 0$, the path satisfies the geodesic equation on U iff it satisfies the geodesic equation on V.

Problem 4: Let S^{n-1} be the sphere in $\mathbb{R}^n$ with radius 1. Let g_S be the induced metric from the standard metric on $\mathbb{R}^3$. (This is called the **round metric**). Let $\gamma : I \to S^{n-1} \to \mathbb{R}^n$be denoted by

$$(x^1(t), \ldots, x^n(t)).$$

1. Compute g_S and the Christoffel symbol Γ^i_{jk}.
2. Show that the geodesic equation is
$$\ddot{x}^i + x^i|\ddot{x}|^2 = 0.$$

Problem 5: Read Appendix 8.1 in Cliff's book about the vector field theorem. You don't need to write down anything for this problem.