

## MATH 230A ASSIGNMENT 6

**Problem 1:** Recall that there is a 1-1 correspondence between vector field  $v$  on a smooth manifold  $M$  and derivation  $\mathcal{L}_v : C^\infty(M; \mathbb{R}) \rightarrow C^\infty(M; \mathbb{R})$ . For two vector fields  $v, w$ , define  $[v, w]$  be the vector field corresponding to the derivation

$$\mathcal{L}_{[v,w]}(f) = \mathcal{L}_v \circ \mathcal{L}_w(f) - \mathcal{L}_w \circ \mathcal{L}_v(f),$$

where  $f \in C^\infty(M; \mathbb{R})$ . This is called the **Lie bracket** on vector fields. Prove the following.

1. For  $a, b \in \mathbb{R}$ ,  $v, w, u$  vector fields,  $[av + bw, u] = a[v, u] + b[w, u]$  and  $[v, aw + bu] = a[v, w] + b[v, u]$ .
2.  $[v, v] = 0$  and  $[v, w] = -[w, v]$ .
3.  $[v, [w, u]] + [u, [v, w]] + [w, [u, v]] = 0$  (Jabobi identity).

**Problem 2:** Suppose  $G$  is a Lie group. Given  $g \in G$ , define  $L_g : G \rightarrow G$  and  $R_g : G \rightarrow G$  by  $L_g(h) = gh$  and  $R_g(h) = hg$ . A vector field  $v$  is called **left-invariant** if  $(L_g)_*v = v$ , **right-invariant** if  $(R_g)_*v = v$ . Let  $\mathfrak{g}$  be the set of left-invariant vector fields. Prove the following.

1.  $\mathfrak{g}$  can be identified with the tangent space  $T_e G$  of  $G$  at the identity  $e$ .
2.  $\mathfrak{g}$  can be identified with the set of right-invariant vector fields.
3. If  $v, w \in \mathfrak{g}$ , then the Lie bracket  $[v, w]$  is also in  $\mathfrak{g}$ .

**Problem 3:** Let  $\pi : P \rightarrow \mathbb{R}$  be a principal  $G$ -bundle over  $\mathbb{R}$ . Prove  $P$  is isomorphic to the product principal bundle  $\mathbb{R} \times G$ .

**Problem 4:** Suppose  $G$  is a subgroup of  $GL(n, \mathbb{R})$  and let  $P = S^1 \times G$  is a product principal bundle over  $S^1$ . Let  $A_0 \in C^\infty(P; \mathfrak{g} \otimes T^*P)$  be the  $\mathfrak{g}$ -valued 1-form which at  $(t, g) \in S^1 \times G$  has the form  $g^{-1}dg$ . Prove the following.

1.  $A_0$  is a connection on  $P$ .
2. any other connection  $A$  on  $P$  must have the form  $A = g^{-1}dg + g^{-1}\alpha g \cdot dt$ , where  $\alpha$  is a map from  $S^1$  to  $\mathfrak{g}$ .

**Problem 5:** Read Chapter 19 in Cliff's book about Hodge star. You don't need to write down anything for this problem.