Class 16 Metric compatible covariant derivative (Chap 15)

In weeks 7-8

① $d : \Omega^k \to \Omega^{k+1}$ $\quad \Omega^k = C^\infty(M; \Lambda^k T^*M)$

space k-forms

$d^2 = 0 \quad H^*_{dR}$

② $\nabla : C^\infty(M; E) \longrightarrow C^\infty(M; E \otimes T^*M)$

vector bundle

pick $v \in TM$ then $\nabla_v : C^\infty(M;E) \circlearrowleft$

③ $d_\nabla : C^\infty(M; E \otimes \Lambda^k T^*M)$

$\to C^\infty(M; E \otimes \Lambda^{k+1} T^*M)$

$d_\nabla s \otimes w \qquad d_\nabla(\sum s_i \otimes w_i) = \sum d_\nabla s \otimes w$

weeks 5-6 we studied metrics on E.

g is section of $E^* \otimes E^*$

and at each fiber it is positive def.

symmetric matrix.

This two weeks. study

metric compatible cov. der

Leibniz rule.

$$d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^{\deg w_1} w_1 \wedge dw_2$$

u, v sections of E.

$g(u,v) : M \longrightarrow \mathbb{R}$ $\quad dg(u,v)$ 1-form

$$dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$$

1-form (metric compatible condition)

M, E. g. ∇

the space of g is convex contractible

the space of ∇ is affine over

$C^\infty(\mathrm{End}(E) \otimes \Lambda^1 T^*U)$

∇^0. $\nabla - \nabla^0$ ψ

$g(\nabla u, v)$ $\nabla u \in C^\infty(E \otimes T^*U)$

$g \in E^* \otimes E^*$ $v \in C^\infty(E)$

$$\nabla u = \sum s_i \, dx^i$$

$$g(\nabla u, v) = \sum_i g(s_i, v) \, dx^i$$

1-form.

$E \otimes E^{*}$.

In local chart U, we have two choices of bases of sections of E.

① Because of the existence of g,
we can choose an orthonormal basis of E

$e_1, \dots, e_n$

$$\boxed{g(e_i, e_j) = \delta_{ij}}$$

no $\frac{\partial}{\partial x^i}$

② In general, $E = TM$. dx^i

there is another basis $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$

$\phi_U : U \longrightarrow \mathbb{R}^n$ $\quad \dim M = n.$

$\varphi_U : TM|_U \longrightarrow U \times \mathbb{R}^n \xrightarrow{\phi_U \times Id} \mathbb{R}^n \times \mathbb{R}^n$

$T\mathbb{R}^n$

depends on the choice of $\left(x^1, \dots, x^n, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$

ϕ_U. but has no relation with g.

In general $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \neq \delta_{ij}$

we will use $\quad g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

$g_{ij} : U \longrightarrow \mathbb{R} \qquad g_{ij} = g_{ji}$

In weeks we write $s_1, \dots, s_n$

for basis of E. but now replace by

$e_1, \dots, e_n$. $v_1, \dots, v_n : U \longrightarrow \mathbb{R}$

Now we change the notation

$s \in C^\infty(E)$

$\nabla s \in C^\infty(E \otimes T^*U)$

$$\nabla s = \sum_{i,j} (\underbrace{ds_i}_{\nabla^0} + \underbrace{\alpha_i^j s_j}_{\alpha}) e_i$$

$\nabla^0 s = ds$ (local .. depends on U)

$\nabla - \nabla^0 = \alpha \in C^\infty(\mathrm{End}(E) \otimes T^*U)$

matrix valued 1-form.

α_i^j is 1-form.

$\alpha = \{\alpha_i^j\}$ is a matrix of 1-form.

In particular.
$$\boxed{\begin{aligned}\nabla e_i &= \sum \alpha_i^j e_j \\ &= \alpha_i^j e_j\end{aligned}}$$

(previously I used α_{ij}).

$(de_i = 0)$

$$\boxed{dg(u,v) = g(\nabla u, v) + g(u, \nabla v)}$$

$g(e_i, e_j) = \delta_{ij} \leftarrow$ Constant functions on U

$dg(e_i, e_j) = 0$

$\quad \|$

$g(\nabla e_i, e_j) + g(e_i, \nabla e_j)$

$$= g(\alpha_i^k e_k, e_j) + g(e_i, \alpha_j^l e_l)$$

I-fu-

$$= \alpha_i^k g(e_k, e_j) + \alpha_j^l g(e_i, e_l)$$

$$= \alpha_i^k \cdot \delta_{kj} + \alpha_j^l \delta_{il}$$

$$= \alpha_i^j + \alpha_j^i$$

$$\alpha = \{\alpha_j^i\}_{ij}$$

$$\alpha^T + \alpha = 0 \qquad \alpha \in C^\infty(\mathrm{End}(E) \otimes T^*M)$$

Claim metric compatible iff $\nabla g = 0$

$\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$

we want to extend ∇ to E^*, $E \otimes E$

for $s^* \in C^\infty(E^*)$ $s \in C^\infty(E)$

$$\boxed{d\langle s, s^*\rangle = \langle \nabla s, s^*\rangle + \langle s, \nabla s^*\rangle}$$

1-form

this is a def of ∇s^*

∇ on $E \otimes E$

$$\nabla(s_1 \otimes s_2) = (\nabla s_1) \otimes s_2 + s_1 \otimes (\nabla s_2).$$

Then we can combine these defs. $\nabla g \Rightarrow$

to get ∇ on $E^* \otimes E^*$ $g \in C^\infty(E^* \otimes E^*)$

$\underline{dg(u,v)}$ $\quad u,v \in C^\infty(E)$

$u\otimes v \in C^\infty(E\otimes E)$

$g \in C^\infty(E^*\otimes E^*)$

$C^\infty((E\otimes E)^*)$

$g(u,v) \to$

$\langle u\otimes v, g\rangle : M \to \mathbb{R}$

$$d\langle u\otimes v, g\rangle = \langle \nabla(u\otimes v), g\rangle + \langle u\otimes v, \nabla g\rangle$$

$$= \langle (\nabla u)\otimes v, g\rangle + \langle u\otimes \nabla v, g\rangle + \langle u\otimes v, \nabla g\rangle$$

$$= g(\nabla u, v) + g(u, \nabla v) + \langle u\otimes v, \nabla g\rangle = 0$$

for any u,v.

$\nabla g \in C^\infty(E^* \otimes E^* \otimes T^*M)$

$\nabla g(u,v) \neq (\nabla g)(u,v)$

$\nabla g = 0 \not\Rightarrow \nabla g(u,v) = 0$ for any u,v

Next: torsion free condition

(only for ∇ on TM)

(because we use d.)

$\nabla \curvearrowright E = TM \qquad \nabla \curvearrowright E^* = T^*M$

$\nabla : C^\infty(M; T^*M) \longrightarrow C^\infty(M; T^*M \otimes T^*M)$

$$\mathcal{A} : C^\infty(M; T^*M \otimes T^*M) \to C^\infty(\Lambda^2 T^*M)$$

$\uparrow$

antisymmetrization map.

by $\mathcal{A}(\omega_1 \otimes \omega_2) = \omega_1 \wedge \omega_2$

$$:= \frac{1}{2}(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1)$$

(In general, we can define

$$\mathcal{A} : C^\infty(T^*M \otimes \Lambda^k T^*M)$$

$$\to C^\infty(\Lambda^{k+1} T^*M)$$

For a 1-form ω

we can define $d\omega$ 2-form. $C^\infty(\Lambda^2 T^*M)$

$$d_\nabla \omega = \nabla \omega \in C^\infty(T^*M \otimes T^*M)$$

$\mathcal{A}(d_\nabla \omega) \in C^\infty(\Lambda^2 T^*M)$

$d\omega \quad \in$

Def the torsion tensor T_∇:

$$C^\infty(T^*M) \to C^\infty(\Lambda^2 T^*M)$$

$$T_\nabla \omega = \mathcal{A}(d_\nabla \omega) - d\omega$$

∇ (on $\underbrace{TM \text{ or } T^*M}$) is called torsion free if $T_\nabla = 0$. i.e.

$$\mathcal{A}(d_\nabla \omega) = d\omega.$$

Rem. In Cliff's book, he wrote

d_∇ for $\mathcal{A}\, d_\nabla$

Thm Given a Riemannian metric on M (i.e. on TM), $\exists$ unique metric compatible, torsion-free covariant derivative, which we call as Levi-Civita connection

∇_{LC}

Rem. we can extend T_∇ to a map: $C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$
$\omega \in C^\infty(\Lambda^k T^*M)$

$d_\nabla \omega \in C^\infty(\Lambda^k T^*M \otimes T^*M)$

$A d_\nabla \omega \in C^\infty(\Lambda^{k+1} T^*M)$

$T_\nabla \omega = \mathbb{A} d_\nabla \omega - d\omega$

Claim: if $T_\nabla \omega = 0$ for 1-form ω, then it vanishes for all k-form ω.

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$$

$$\mathbb{A} d_\nabla(\omega_1 \wedge \omega_2) = \mathbb{A} d_\nabla \omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \mathbb{A} d_\nabla \omega_2$$

d only depends on Ω^k (only for M)

d_∇ depends on ∇ $\quad \mathbb{A}$

$A\, d_g\, w = dw$

for $M + \nabla$ only for M

the metric compatible connection

We start with M with a metric g on $E = TM$.

(M, g) is called a Riemannian manifold

Then $\{\nabla g = 0,\ T_\nabla = 0\}$ can be regarded as equations on the space of ∇. $C^\infty(\mathrm{End}(E) \otimes T^*M)$

Later. (the final goal of this course)

is to understand the Yang-Mills equation

$$F_{\nabla}^{\ast} = 0$$ ← introduced later

this is also an equation on the space of ∇.

∇_{LC} depends uniquely by (M, g)

Sketch the idea for the next class.

There are two proofs

① using orthonormal basis $g(e_i, e_j) = \delta_{ij}$

② using coordinate basis $\frac{\partial}{\partial x^1}, \cdots \frac{\partial}{\partial x^n}$

Warm-up computation (everything locally)

$$\nabla e_i = \sum_j e_j \otimes \alpha^j_i$$

$e_1, \dots, e_n$ is orthonormal basis of

$E = TM$

Question: what is the formula for

∇ on T^*M $(= E^*)$

e^i as dual basis on E^*

$$\langle e_i, e^j \rangle = \delta_{ij}$$

$$\nabla e^i = \sum_j e^j \otimes \beta^i_j$$

$$0 = d\langle e_i, e^j\rangle = \langle \nabla e_i, e^j\rangle$$

$$\underbrace{}_{\delta_{ij}} \quad + \langle e_i, \nabla e^j\rangle$$

$$= \langle \alpha_i^k e_k, e^j\rangle + \langle e_i, e^l \otimes \beta_l^j\rangle$$

$$= \alpha_i^j + \beta_i^j \longrightarrow \beta_i^j = -\alpha_i^j$$

$$\nabla g = 0 \Longrightarrow \alpha_i^j + \alpha_j^i = 0$$

$$\Longrightarrow \beta_i^j + \beta_j^i = 0$$

$$d(\omega_1 \wedge \omega_2) = \cdots$$