Class 17 Levi-Civita Connection

M manifold g metric on TM

(M,g) ∇ on TM (which also induced ∇ on T^*M)

∇ is compatible with g : if

1) $dg(u,v) = g(\nabla u, v) + g(u, \nabla v)$

2) $\underline{\nabla g = 0}$ $g \in C^\infty(\underline{T^*M \otimes T^*M})$

$\hookrightarrow$ as an equation on the space of ∇

$\longleftrightarrow$ $\underline{C^\infty(\mathrm{End}(TM) \otimes T^*M)}$

E

∇ torsion free if $T_\nabla = 0$

$T_\nabla = \mathbb{A}\nabla - d : C^\infty(T^*M) \to C^\infty(\Lambda^2 T^*M)$

$$d : C^\infty(T^*M) \to C^\infty(\Lambda^2 T^*M)$$

$$\nabla : C^\infty(T^*M) \to C^\infty(T^*M \otimes T^*M)$$

$$\underline{A} : C^\infty(\underline{T^*M \otimes T^*M}) \to C^\infty(\Lambda^2 T^*M)$$

$$\omega_1 \otimes \omega_2 \longmapsto \omega_1 \wedge \omega_2$$

$$\underline{\underline{d_\nabla}} : C^\infty(E \otimes \Lambda^k T^*M) \to C^\infty(E \otimes \Lambda^{k+1} T^*M)$$

$$\nabla \qquad d \qquad E = TM$$

$$A\nabla : C^\infty(T^*M) \to C^\infty(\Lambda^2 T^*M)$$

$$E \otimes T^*M \quad ?? \quad A$$

$$A\nabla : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$$

$$\nabla : C^\infty(\Lambda^k T^*M) \to C^\infty(\underline{\underline{\Lambda^k T^*M \otimes T^*M}})$$

$T_\nabla = A\nabla - d$ in this case

Thm. Given (M, g). $\exists$ unique ∇

satisfies $\begin{cases} \nabla g = 0 \longrightarrow \text{metric compatible} \\ T_\nabla = 0 \longrightarrow \text{torsion free} \end{cases}$

this ∇ is called Levi-Civita connection

Geodesic equation (locally) $\gamma : I \to M$

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

$\gamma_* \frac{\partial}{\partial t} \in C^\infty(TM)$

$$\Longleftrightarrow \nabla^{LC}_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t} = 0$$

related to parallel transportation

Before proof. we notice now $E = TM$

there are two "natural" basis of sections

locally. (on chart U)

① orthonormal basis $e_1, \cdots, e_n$

dual basis $e^1, \cdots, e^n$ for T^*M

$$g(e_i, e_j) = \delta_{ij} \Rightarrow d\, g(e_i, e_j) = 0$$

② coordinate basis $\overbrace{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}}^{}$

dual basis $dx^1, \cdots, dx^n$ $\qquad TR^n$

$$\left(\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta_{ij} \right) \qquad T^*R^n$$

$$\phi_U : U \to \mathbb{R}^n \qquad \overbrace{\qquad\qquad}^{\text{coordinate map}}$$

$$\varphi_U : TU \to T\mathbb{R}^n \quad x^1, \cdots, x^n, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$$

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} : M \to \mathbb{R}$$

Pf 1 (using ①)

assume

$$\nabla e_i = \sum_j e_j \otimes \alpha_i^j = \sum_j \alpha_i^j e_j$$

$$= \alpha_i^j e_j$$ (j appears twice we take the sum)

α_i^j is 1-form.

$\alpha = \{\alpha_i^j\}$ is a matrix-valued 1-form.

$$\nabla g = 0 \iff \alpha_i^j = -\alpha_j^i$$

assume

$$\nabla e^i = e^j \otimes \beta_j^i$$ β_j^i 1-forms

$$0 = d\langle e_i, e^j \rangle = \cdots$$

$$= \alpha_i^j + \beta_i^j \qquad \alpha = -\beta^T$$

$$\Rightarrow \nabla g = 0 \iff \beta_j^i = -\beta_i^j$$

$$\underline{\mathbb{A}\nabla e^i} = \mathbb{A}(e^j \otimes \beta^i_j)$$

$$= \mathbb{A}(\beta^i_{jk}\, e^j \otimes e^k)$$

$$\left(\beta^i_j = \beta^i_{jk}\, e^k \qquad \beta^i_{jk} : M \to \mathbb{R}\right)$$

$$= \beta^i_{jk}\, e^j \wedge e^k = \beta^i_{jk}\frac{1}{2}(e^j \otimes e^k - e^k \otimes e^j)$$

$$= \sum_{j,k} \frac{1}{2}(\beta^i_{jk} - \beta^i_{kj})\, e^j \otimes e^k$$

assume

$$de^i = \sum_{j<k} \gamma^i_{jk}\, e^j \wedge e^k \qquad \Lambda^2$$

$$= \sum_{j<k} \gamma^i_{jk} \frac{1}{2}(e^j \otimes e^k - e^k \otimes e^j)$$

$$= \sum_{j,k} \frac{1}{2}\gamma^i_{jk}\, e^j \otimes e^k$$

Set up

$$\boxed{\gamma^i_{jk} = -\gamma^i_{kj}}$$

$$T_\nabla = 0 \Rightarrow \cancel{A}\nabla e^i - de^i = 0$$

$$\Rightarrow \frac{1}{2}\left(\beta^i_{jk} - \beta^i_{kj} - \gamma^i_{jk}\right) = 0 \quad (2)$$

$$\underbrace{\beta^i_j = -\beta^j_i}_{\text{1-forms}} \Rightarrow \underbrace{\beta^i_{jk} = -\beta^j_{ik}}_{\substack{\text{functions} \\ \text{0-forms}}} \quad (1)$$

change indices

$$\beta^j_{ki} - \beta^j_{ik} - \gamma^j_{ki} = 0 \quad (2')$$

$$\beta^k_{ij} - \beta^k_{ji} - \gamma^k_{ij} = 0 \quad (2'')$$

$(2) + (2') - (2'')$

$$\Rightarrow \beta^i_{jk} = \frac{1}{2}\left(\gamma^i_{jk} + \gamma^j_{ki} - \gamma^k_{ij}\right)$$

If we have another solution $(\beta')^i_{jk}$

Suppose $\eta^i_{jk} = \beta^i_{jk} - (\beta')^i_{jk}$

then (2) $\Rightarrow$ $\eta^i_{jk} - \eta^i_{kj} = 0$

(because $\gamma^i_{jk} - \gamma^i_{jk} = 0$)

(1) $\longrightarrow$ $\eta^j_{ik} = -\eta^i_{jk}$

Claim $\eta^i_{jk} = 0$ so solution is unique

$$\eta^j_{ik} \overset{(1)}{=} -\eta^i_{jk} \overset{(2)}{=} -\eta^i_{kj} \overset{(1)}{=} \eta^k_{ij}$$

$$\overset{(2)}{=} \eta^k_{ji} \overset{(1)}{=} -\eta^j_{ki} \overset{(2)}{=} -\eta^j_{ik}$$

Pf 2. using $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$

$d(dx^i) = 0$ so torsion free condition α is easier.

Suppose $\nabla \frac{\partial}{\partial x^i} = \Gamma^j_{ik} \frac{\partial}{\partial x^j} \otimes dx^k$

$\nabla dx^i = -\Gamma^i_{jk} dx^j \otimes dx^k$

β.

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl} (\partial_i g_{lk} + \partial_k g_{il} - \partial_l g_{ik})$$

Christoffel symbol

$$T_\nabla = 0 \overset{Q:}{\Longrightarrow} \Gamma^i_{jk} = \Gamma^i_{kj}$$

$$(\gamma^i_{jk} = 0) \quad \beta^i_{jk} = -\Gamma^i_{jk}$$

$$\nabla g = 0 \qquad g = g_{ij}\, dx^i \otimes dx^j$$

$$g_{ij} = g_{ji} : U \to \mathbb{R}$$

Define $\{g^{ij}\} = \{g_{ij}\}^{-1}$ matrix

$$g^{ij} : U \to \mathbb{R}$$

$$0 = \nabla g = \nabla(g_{ij}\, dx^i \otimes dx^j)$$

$$= \underline{(\nabla g_{ij})}\, dx^i \otimes dx^j + g_{ij} \underline{(\nabla dx^i)} \otimes dx^j$$

$$g_{ij}\, dx^i \otimes \underline{(\nabla dx^j)}$$

∇g_{ij} is a 1-form

$$\nabla_l g_{ij} = \frac{\partial g_{ij}}{\partial x^l}$$

$$= \partial_l g_{ij} : U \to \mathbb{R}$$

$$\nabla_l = \nabla_{\frac{\partial}{\partial x^l}}$$

$$\nabla_l dx^i = -\Gamma^i_{lk} dx^k$$

$$0 = (\nabla_l g_{ij}) dx^i \otimes dx^j + g_{ij} \nabla_l dx^i \otimes \ldots$$

$$= \partial_l g_{ij} dx^i \otimes dx^j$$

$$- g_{ij} (\Gamma^i_{lp} dx^p \otimes dx^j + \Gamma^j_{lq} dx^i \otimes dx^q)$$

change i, j, k to $j\,k\,i$, $k\,i\,j$

$$g_{ij}\,\Gamma^i_{lp} = \frac{1}{2} g_{ij}\, g^{ir}\left(\partial_l g_{rp} + \partial_p g_{lr} - \partial_r g_{lp}\right)$$

$$g_{ij}\, g^{ir} = \delta^r_j$$

Recall $\dfrac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk}\dfrac{d\gamma^j}{dt}\dfrac{d\gamma^k}{dt} = 0$

or $\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0$

$\dot{\gamma} = \gamma_* \dfrac{\partial}{\partial t}$

$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$

Def. Let $\gamma : [0,1] \to M$ is a path

a section s of $E|\gamma = \pi^{-1}(\gamma)$ is

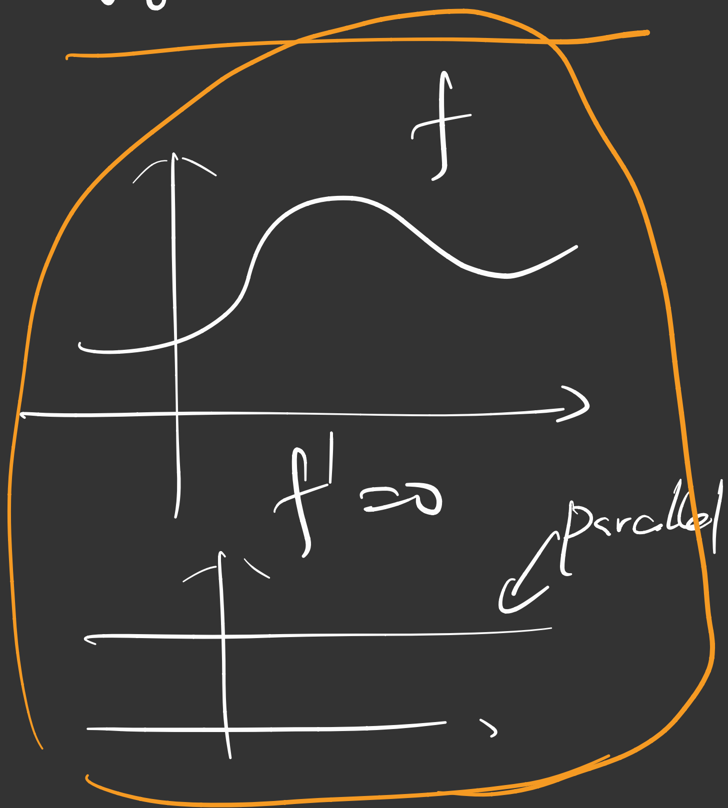
parallel if $\nabla_{\dot\gamma} s = \langle \gamma^* \frac{\partial}{\partial t}, \nabla s \rangle$

vanishes.

$\nabla : C^\infty(E) \to C^\infty(E \otimes T^*M)$

$\gamma \quad \nabla_{\dot\gamma} : C^\infty(E) \to C^\infty(E)$

$\nabla_{\dot\gamma} s = 0$



write $\nabla s = (ds_i + \alpha^{ij} s_j) e_i$

where e_i are basis of E

(no orthonormal condition because no g)

$$\gamma^* \frac{\partial}{\partial t} = \frac{d\gamma^k}{dt}\frac{\partial}{\partial x^k}$$

$$\alpha^{ij} = \alpha^{ij}_k dx^k \qquad \alpha^{ij}_k : U \to R$$

$$\nabla_{\dot\gamma} s = \sum_i \left(\frac{d(s_i \circ \gamma)}{dt} + \alpha^{ij}_k \frac{d\gamma^k}{dt} s_j \circ \gamma \right) e_i$$

$$\langle ds_i, \gamma^* \frac{\partial}{\partial t} \rangle$$

$\nabla_{\dot\gamma} s = 0$ iff $\langle \nabla_{\dot\gamma} s, e^i \rangle = 0$

e^i dual basis

Take $s = \dot{\gamma} = \frac{d\gamma}{dt}$ $\nabla = \nabla_{LC}$

then $s_i \circ \gamma = \frac{d\gamma^i}{dt}$

$$\nabla_{\frac{\partial}{\partial x^l}} = \left(d\cancel{\left(\frac{\partial}{\partial x^l}\right)_i}^{\,0} + \alpha^{ij}\left(\frac{\partial}{\partial x^l}\right)_j \right) \frac{\partial}{\partial x^i}$$

$$= \alpha^{il} \frac{\partial}{\partial x^i} \qquad \alpha^{il}_k = \Gamma^l_{ik}$$

$$\left(\left(\frac{\partial}{\partial x^l}\right)_i = \delta_{il} \right)$$

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \iff \frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^k}{dt} \frac{d\gamma^j}{dt} = 0$$