Math 230a: Differential geometry

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Course Page: http://scholar.harvard.edu/fanye/classes/math230a-differential-geometry

Main Reference: Differential geometry by Clifford H. Taubes Chap 1–16

Office Hours: Fan Tuesday 1:30 – 2:30 pm Science Center 505H
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More information: see syllabus on the page.

Style of this course:

1. This is like a tour to a garden or a zoo, which means you will meet many new things but can't understand them well at the first time. Don't be afraid to discuss with classmates about basic definitions and examples, and go back to previous materials once and once again.

2. I study low-dimensional topology, so I will rearrange the materials in the book in a topologist way. That means I will focus more on the construction and motivation. The book contains many useful calculations. It's good to read and calculate by yourself.

3. Topics: Manifolds, Lie groups, vector bundles, metrics, geodesics, connections, curvatures, principal bundles, Yang–Mills equations, ...

Class 1. Introduction to manifolds (Clift Chap 1)

Def 1. A topological manifold of dimension n is a paracompact, Hausdorff topological space s.t. each point (pt) has a neighborhood (nbhd) that is homeomorphic to $\mathbb{R}^n$

Explanation for red lines:

- topological space: a set X with distinguished collection $\mathcal{O}$ of subsets of X called open sets s.t.
  1) $\phi, X \in \mathcal{O}$
  2) Any union of sets in $\mathcal{O}$ is still in $\mathcal{O}$
  3) finite intersections of sets in $\mathcal{O}$ is still in $\mathcal{O}$

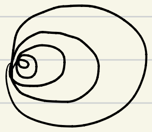
Ex: The standard topology on $\mathbb{R}^n$: $\mathcal{O}$ is the collection of sets given by unions and finite intersections of open balls

$$\{x \mid |x-p| < r \text{ for } p \in \mathbb{R}^n, r \in \mathbb{R}_+\}$$

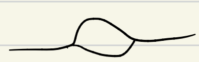
- paracompact: every open cover has a locally finite ~~subcover~~ refinement
- locally finite: each pt is in finitely many open sets.
- Hausdorff: any two pts have disjoint, open nbhds

These two conditions exclude some "wild" topo space:

1) Union of rays with irrational slopes 

2) Hawaiian ring  $\bigcup_{n \in \mathbb{N}} \{(x,y) \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}$

(infinite union)

3) Bug-eyed line $\mathbb{R} \cup \mathbb{R} / x \sim y$ if $x = y \neq 0$ 

4) The line of irrational slope in $\mathbb{R}^2/(x,y) \sim (x+m, y+n)$ $m, n \in \mathbb{Z}$

- nbhd of pt: an open set containing the pt
- homeomorphism: A continuous, 1-1 map with continuous inv
- Continuous: the preimage of an open set is an open set.

Usually we add the connected condition to a topo mfd

- Connected: it is not a disjoint union of two open sets

Fact: $\mathbb{R}^n$ is not homeo to $\mathbb{R}^m$ if $m \neq n$

So dim of a topo mfd is a definite number.

Ex of topo mfds:

1) $\mathbb{R}^n$, any open subset in $\mathbb{R}^n$

2) $S^n = \{ x \in \mathbb{R}^n \mid |x| = 1 \}$

3) $T^n = \mathbb{R}^n / (x_1, \cdots, x_n) \sim (x_1 + m_1, \cdots, x_n + m_n)$, $m_1, \cdots, m_n \in \mathbb{Z}$.

4) product of manifolds. Indeed $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ copie}}$

Then we move to smooth manifolds.

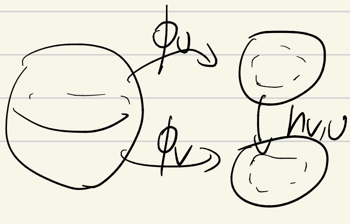
Def 2. For a topo mfd X, the nbhd of a pt is called a (coordinate) chart. A collection of charts that covers X is called a (coordinate) atlas.

If U, V are two charts of X. $U \cap V \neq 0$, then

$\exists\ \phi_U : U \to \mathbb{R}^n$, $\phi_V : V \to \mathbb{R}^n$ homeos

$$h_{V,U} = \phi_V \circ \phi_U^{-1} : \underset{\mathbb{R}^n}{\phi_U(U \cap V)} \subset \to \underset{\mathbb{R}^n}{\phi_V(U \cap V)} \subset \text{ is also a homeo}$$

It is called the (coordinate) transition function.



Given an atlas $\mathcal{U}$ and all trans functions $h_{V,U}$.

we can recover X by the quotient of the disjoint union

$$\coprod_{U \in \mathcal{U}} \mathbb{R}^n|_U \Big/ x \in \mathbb{R}^n|_U \sim h_{VU}(x) \in \mathbb{R}^n|_V, \ \forall U, V \in \mathcal{U}$$

Idea: • Given the atlas and transition functions

we can always do computations locally

• We can understand the same manifold by different atlas

Def 3: An atlas is called a <u>smooth structure</u> if all transition functions are smooth (infinitely differentiable) or in C^∞

A topo mfd with a smooth str is called a <u>smooth mfd</u>

Two mfds are <u>diffeomorphic</u> if $\exists$ a smooth map btw them with a smooth inverse

Then we can do calculus on smooth manifolds

Fact. Smooth str is not unique for some topo mfds.

does not exist for some topo mfds.

For dimension n mfd,

If $n = 0, 1, 2, 3$, then we have existence and uniqueness of the smooth str.

If $n \geq 4$. there exist counterexamples

Two basic examples :

$\mathbb{R}^n$: $\exists$ unique smooth str if $n \neq 4$.

uncountably many if $n = 4$.

S^n : $\exists$ unique smooth str if

$n = 1, 2, 3, 5, 6, 12, 56, 61$

non unique if $n \geq 7$ $\neq 56, 61$ odd

$n = 7$ $\exists$ 28 smooth str. (Milnor)

$n \geq 8$ < 140 even

widely open : $n = 4$

3 : Poincaré conj Solved by Grigori Perelman (2002)

5, 6, 12 : Michel A. Kervaire and John W. Milnor (1963)

56 : Recent work of Daniel C. Isaksen (2019)

61 : Recent work of Guozhen Wang and Zhouli Xu (2017)

$\underbrace{1, 2, 3, 4,}_{\text{low-dimensional topology}} \underbrace{5, 6, 7, 8, \cdots\cdots}_{\text{algebraic topology}}$

even : Complex geometry, symplectic geometry,

Kähler geometry, $\cdots$

odd : contact geometry, $\cdots$

Class 2: Smooth manifolds (Chap 1)

Review: (some people use second countable, i.e. countable topo basis
this is equivalent to paracompact with countably many connected components)

Def 1. A topo mfd of dim n is a paracompact Hausdorff topo space s.t. each pt has a nbhd that is homeo to $\mathbb{R}^n$. such nbhd is called a chart

The collection of charts is called an atlas.

For two charts U, V. $\exists$ transition function

$$h_{V,U}: \underset{\mathbb{R}^n}{\underset{\cap}{\phi_U(U \cap V)}} \longrightarrow \underset{\mathbb{R}^n}{\underset{\cap}{\phi_V(U \cap V)}}$$

If $h_{V,U}$ is smooth (infinitely differentiable),
then the atlas is called a smooth str.

A topo mfd with an smooth str is called a smooth mfd.

Rem. $\exists$ topo mfd that is not smoothable
$\exists$ topo mfd that has non unique smooth str

Def 2. Suppose M and N are smooth mfds.
Let $\mathcal{U}$ and $\mathcal{V}$ be locally finite atlases corresponding to smooth str of M and N (we use the paracompact condition)
A map $h: M \to N$ is called smooth if each map in
$\{\phi_V \circ h \circ \phi_U^{-1}: h(U) \cap V \neq \emptyset\}_{(U,\phi_U) \in \mathcal{U},\, (V,\phi_V) \in \mathcal{V}}$
is smooth

Ex: Consider the manifold $\mathbb{R}'$ with only one chart $\phi_1 : \mathbb{R}' \xrightarrow{x^3} \mathbb{R}$

and also the manifold $\mathbb{R}''$ with only one chart $\phi_2 : \mathbb{R}'' \xrightarrow{x} \mathbb{R}$

Define $f : \mathbb{R}' \xrightarrow{x^3} \mathbb{R}''$, then $\phi_2 \circ f \circ \phi_1^{-1} = x : \mathbb{R} \to \mathbb{R}$, f, g are smooth inverses

$g : \mathbb{R}'' \xrightarrow{x^{\frac{1}{3}}} \mathbb{R}'$. $\phi_1^{-1} \circ g \circ \phi_2 = x : \mathbb{R} \to \mathbb{R}$ $\quad \mathbb{R}'$ is diffeo to $\mathbb{R}''$

The reason to define the smooth str is to do calculus on mfds

Thm 1 (Inverse function thm)

Let $U \subset \mathbb{R}^n$ be an open set and let $\psi : U \to \mathbb{R}^n$ be a smooth map. Choose a pt $p \in U$. Let ψ_* be the matrix $\left\{ \frac{\partial \psi^j}{\partial x^i} \right\}_{i,j \in [1,n]}$. It is called the <u>Jacobian</u>

If ψ_* is invertible, then $\exists$ nbhd $V \subset \mathbb{R}^n$ of $\psi(p)$ and a smooth map $G : V \to U$ s.t. $G(\psi(p)) = p$ and

- $G \circ \psi = Id$ on nbhd $U' \subset U$ of p
- $\psi \circ G = Id$ on V

i.e. ψ has a local inverse around p.

Idea: The smoothness assumption reduces a non-linear question to a linear one (first derivative). We will use Taylor's thm with remainder to prove it

Pf: $\psi(x) = \psi(p) + (\psi_*|_p)(x-p) + \underbrace{R(x-p)}_{\text{Remainder}}$

$$|R(u)| \le c|u|^2 \quad |R(u) - R(v)| \le c|u-v|(|u|+|v|)$$

Set $u = x - p = \psi_*^{-1}(\psi(x) - \psi(p) - R(u))$

Set $\psi(x) = q$. We need to find a pt

$u_q = \psi_*^{-1}(q - \psi(p) - R(u_q))$ if q is near $\psi(p)$

We introduce contraction mapping thm to find u_q

Lem 1. Let $r>0$ $\delta\in(0,1)$. $B=\{x\in\mathbb{R}^n \mid |x|<r\}$

If $f: B \to B$ satisfies

1) $|f(x)|<\delta r \quad \forall x\in B$

2) $|f(x)-f(y)| \le \delta|x-y| \quad \forall x,y\in B$

then $\exists$ unique $p\in B$ s.t. $f(p)=p$

Pf: Take any $x_0\in B$. Set $x_{n+1}=f(x_n)$.

Since $|x_n-x_m|<\delta^{n-m} r$ for $n>m$, we know

$\{x_n\}_{n\ge0}$ is a Cauchy sequence, which converges in B

Let p be the limit, then $f(p)=p$ $(f(x_n)=x_{n+1})$

If q also satisfies $f(q)=q$. then

$$|p-q|=|f(p)-f(q)|\le\delta|p-q| \Rightarrow p=q. \qquad \square$$

We come back to the pf of inverse function thm.

Take $f(u)=\mathcal{F}_*^{-1}(q-\mathcal{F}(p)-R(u))$

Then f maps a small ball to itself because $|R(u)|\le c|u|^2$

$$\left(|u|\le\frac{1}{2c},\ |R(u)|\le\frac{1}{4c},\ \delta=\frac{1}{2}\right)$$

$$|f(u)-f(v)|\le|R(u)-R(v)|\le c|u-v|(|u|+|v|)<\delta|u-v|$$

when $|u|,|v|$ are small

We apply Lem 1 to obtain the unique element u_q s.t. $f(u_q)=u_q$

We can also use Lem 2 to analyze the function $q\mapsto u_q$.

to show it is smooth; see A 1.1.1 of Cliff's book.

Def 3: Let $U \subset \mathbb{R}^n$ be an open set. $\psi: U \to \mathbb{R}^m$ a smooth map

$a \in \mathbb{R}$ is a <u>regular value</u> of ψ if the Jacobian ψ_* is surjective at any pt in $\psi^{-1}(a)$

Thm 2: (Sard thm) Let U, ψ be defined in Def 3. Then the set of regular values of ψ have full measure.

Ex: $\psi: \mathbb{R}^n \to \mathbb{R} \quad (x_1, \cdots, x_n) \to \sum_{i=1}^{n} x_i^2$

$$\psi_* = \left(\frac{\partial \psi}{\partial x_1}, \cdots, \frac{\partial \psi}{\partial x_n}\right) = (2x_1, \cdots, 2x_n)$$

$\forall\, r > 0$ is a regular value

Thm 3 (Implicit function thm) Let $U \subset \mathbb{R}^n$; $\psi: U \to \mathbb{R}^{n-m}$ $n-m \geq 0$ and a is a regular value. Then $\psi^{-1}(a) \subset U$ is a smooth mfd of dim m. For $p \in \psi^{-1}(a)$, $\exists$ a ball $B \in \mathbb{R}^n$ with center p and a diffeo $\phi: B \to \mathbb{R}^n$

s.t. $\phi(B \cap \psi^{-1}(a)) = \{x_{m+1} = \cdots = x_n = 0\} \cap B_\varepsilon(0)$ for some $\varepsilon > 0$

Pf: Let $v_1, \cdots, v_{n-m}$ span $\ker \psi_*|_p$ and let π be the projection: $\mathbb{R}^n \to \ker \psi_*|_p \cong \mathbb{R}^m$

Define $\phi: B \to \mathbb{R}^n \quad \phi(x) = (\psi(x), \pi(x))$

Then $\phi_*|_p$ is invertible and we apply the inverse function thm to find some local inverse G

Implicit function thm is a good way to construct smooth mfds.

Def 4: A <u>submfd</u> M in $\mathbb{R}^n$ is a subset s.t. $\forall\, p \in M$ $\exists\, U_p \subset \mathbb{R}^n$ and $\psi_p : U_p \to \mathbb{R}^{n-m}$ with 0 as a regular value and $M \cap U_p = \psi_p^{-1}(0)$

Suppose N is a smooth mfd of dim n. A subset $M \subset N$ is a <u>submfd</u> if for any $p \in M$, $\exists\, U_p \subset N$
$\phi_p : U_p \to \mathbb{R}^n$ s.t. $\phi_p(M \cap U_p)$ is a submfd of $\mathbb{R}^n$.

Cor: Let $f : N \to M$ be a smooth map. Let $p \in N$. If for any $q \in f^{-1}(p)$, any chart $U_q \subset N$ with $\phi_q : U_q \to \mathbb{R}^n$, any chart $U_p \subset M$ with $\phi_p : U_p \to \mathbb{R}^m$, the Jacobian of $\phi_p \circ f \circ \phi_q^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ at $\phi_q(q)$ is surjective, then $f^{-1}(p)$ is a smooth submfd of N.

Fact: For any smooth mfd M, $\exists\, \phi : M \to \mathbb{R}^N$ $N \gg 0$ s.t. $\phi(M)$ is a submfd of $\mathbb{R}^N$, and ϕ is a diffeomorphism.

Such ϕ is called an embedding.

The proof use a partition of unity. (see Appendix 1.2)

Class 3 Lie group (Chap 2)

Roughly speaking, a Lie group is both a group and a smooth mfd.

Def 1. A group G is a set with a multiplication law:

$m: G \times G \to G$ s.t. (we write $m(a,b)$ as ab.)

1) $a(bc) = (ab)c$

2) $\exists\, e \in G$ s.t. $ae = ea$ for $\forall\, a \in G$ called identity

3) Any element $a \in G$ has an inverse $a^{-1} \in G$ s.t.

$$aa^{-1} = a^{-1}a = e.$$

Def 2. A Lie group G is a smooth manifold with a group multiplication s.t. the maps

$m: G \times G \to G$ and $a \mapsto a^{-1}$ are smooth.

this condition can be deduced from smoothness of m.

- $M(n, \mathbb{R})$: the space of $n \times n$ matrices

as a manifold, we have $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$.

But the multiplications are different.

On $M(n,\mathbb{R})$, it is the matrix multiplication

On $\mathbb{R}^{n^2}$, we can use the addition of vectors to define a multiplication

However, $M(n,\mathbb{R})$ is not a Lie group because the multiplication is not invertible

- $GL(n,\mathbb{R})$: general linear group

  the space of invertible matrices in $M(n,\mathbb{R})$

  i.e. $\det A \neq 0$ for $A \in GL(n,\mathbb{R})$

This is an open subset of $M(n,\mathbb{R})$ because

$\det : M(n,\mathbb{R}) \to \mathbb{R}$ is a smooth function.

$GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R}-\{0\})$, where $\mathbb{R}-\{0\}$ is open.

So $GL(n,\mathbb{R})$ is a smooth mfd.

(Moreover, the matrix multiplication is invertible on $GL(n,\mathbb{R})$)

So $GL(n,\mathbb{R})$ is a Lie group. $e = Id_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

Def 3: A subgp H of a gp G is a subset containing the indenty e, s.t. $m(a,b) \in H$ for $a,b \in H$

Lem 1. A subgroup of a Lie group that is also a submanifold is a Lie group with respect to the induced smooth str.

Pf: The restrictions of smooth maps $m : G \times G \to G$ and $a \mapsto a^{-1}$ to a submfd H are also smooth maps $\square$

Ex.

- $SL(n,\mathbb{R})$: Special linear group

  the space of matrices with $\det = 1$

  Note that $\det(AB) = \det(A)\det(B)$

We can show $\det_* |_1 \neq 0$

So by Implicit function thm
$SL(n,\mathbb{R})$ is a smooth mfd, and hence a Lie group.
Since det is a map to $\mathbb{R}$, we only need to find one direction s.t. the partial derivative is nonzero

Consider $t \longmapsto tA$ for $t \in \mathbb{R}$ $\det(tA) = t^n \det(A)$

$$\partial_t(\det(tA)) = nt^{n-1}\det(A) \xrightarrow[\det(A)=1]{t=1} n \neq 0.$$

More examples

- $O(n)$ orthogonal group.

$$\{A \in GL(n,\mathbb{R}) \mid A^{-1} = A^T\}$$

Note $\det(A^{-1}) = (\det(A))^{-1}$

$\det(A^T) = \det(A)$

So $\det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$ for $A \in O(n)$

- $SO(n)$: Special orthogonal group

$$\{A \in O(n) \mid \det(A) = 1\}$$

$A \in SO(n)$ $A \cdot \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$ in the other component of $O(n)$

So two components of $O(n)$ are diffeomorphic if smooth.

To prove $SO(n)$ is smooth, we use implicit function thm again

$F: GL(n,\mathbb{R}) \longrightarrow Sym(n) = \{n \times n \text{ symmetric matrices}\}$

$F(A) = AA^T$. is the Jacobian F_* surjective at $F^{-1}(Id)$?

Suppose $F(A_0) = Id$ $F_*|_{A_0}(A) = A_0A^T + AA_0^T$

Let S be any matrix in $Sym(n)$. set $A = \frac{1}{2}SA_0$

$F_*|_{A_0}(A) = \frac{1}{2}(A_0A_0^TS + SA_0A_0^T) = S$ because $A_0A_0^T = Id$

More things are introduced in the course of.

Lie group, Lie algebra, and their representations

Complex matrix groups.

- $M(n, \mathbb{C})$. the space of $n \times n$ complex matrices

$M(n, \mathbb{C}) \cong \mathbb{R}^{2n^2}$ because a complex number

$$c = a + bi \quad a, b \in \mathbb{R}$$

- $GL(n, \mathbb{C})$ is the open set in $M(n, \mathbb{C})$ where the matrix is invertible, i.e. $\det \neq 0$

It is a Lie group

Let $C \in GL(n, \mathbb{C})$ $\quad C = A + Bi \quad A, B \in GL(n, \mathbb{R})$

Then C can be viewed as an element in $GL(2n, \mathbb{R})$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

Indeed, let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

$$GL(n, \mathbb{C}) = \{ M \in GL(2n, \mathbb{R}) \mid MJ = JM \}$$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix}$$

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix}$$

- $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid \det A = 1\}$
- $U(n) = \{A \in GL(n,\mathbb{C}) \mid A^*A = Id\}$ $\quad A^* = \bar{A}^T$
- $SU(n) = \{A \in U(n) \mid \det(A) = 1\}$ $\quad |\det(A)| = 1$

We still prove they are smooth mfds by implicit function thm

$SL(n,\mathbb{C})$ similar to $SL(n,\mathbb{R})$

Need to show $\det_*|_A$ is surjective. for $\det A = 1$

$t = x + yi \in \mathbb{C}$ $\quad \partial_x \det(tA) = 2n\, t^{2n-1} \det A$

$\quad \partial_y \det(tA) = i\, 2n t^{2n-1} \det A$

(Note that $tA \in M(n,\mathbb{C})$ because det is linear from $M(n,\mathbb{C})$ to $\mathbb{C}$)

$U(n)$. Define $F: GL(n,\mathbb{C}) \to Herm(n) = \{H \in M(n,\mathbb{C}) \mid H^* = H\}$

$F(A) = A^*A$ $\quad$ For $A_0 \in F^{-1}(Id)$

$F_*|_{A_0}(A) = A^*A_0 + A_0^*A$

Give $H \in Herm(n)$ take $A = \frac{1}{2}A_0H$

Then $F_*|_{A_0}(A) = \frac{1}{2}(HA_0^*A_0 + A_0^*A_0H) = H$.

$SO(n)$: Since $|\det A| = 1$ for $A \in U(n)$ $\det A = e^{i\theta_0}$

Consider $\arg\det A = \theta_0$

$\arg(\det(e^{i\theta}A)) = n\theta + \theta_0$ $\quad \partial_\theta \arg(\det(e^{i\theta}A)) = n$

so the Jacobian of $\arg\det : U(n) \to S^1$ is surjective